On the computation of the noncentral *F* and noncentral beta distribution

Ali Baharev, Sándor Kemény
Budapest University of Technology and Economics
Department of Chemical and Environmental Process Engineering
Budapest, Hungary, H-1521
ali.baharev@gmail.com, kemeny@mail.bme.hu

This is a preprint of an article appeared in
Statistics and Computing, 2008, 18 (3), 333-340

The original publication is available at http://www.springerlink.com.
DOI 10.1007/s11222-008-9061-3
http://dx.doi.org/10.1007/s11222-008-9061-3

Ali Baharev is PhD student at the Budapest University of Technology and Economics, Department of Chemical and Environmental Process Engineering, Budapest, Hungary, H-1521 (email: ali.baharev@gmail.com); and Sándor Kemény is professor at the same institution (email: kemeny@mail.bme.hu). The authors wish to thank Mr. Richárd Király for his preliminary work. The authors are grateful to the Associate Editor of STCO and the unknown reviewers for their helpful suggestions.

ABSTRACT AND KEY WORDS

Unfortunately many of the numerous algorithms for computing the cdf and noncentrality parameter of the noncentral *F* and beta distributions can return completely incorrect results as demonstrated in the paper by examples. Existing algorithms are scrutinized and those parts that involve numerical difficulties are identified. As a result, a pseudo code is presented in which all the known numerical problems are resolved. This pseudo code can be easily implemented in programming language C or FORTRAN without understanding the complicated mathematical background.
Symbolic evaluation of a finite and closed formula is proposed to compute exact cdf values. This approach makes it possible to check quickly and reliably the values returned by professional statistical packages over an extraordinarily wide parameter range without any programming knowledge.
This research was motivated by the fact that a most useful table for calculating the size of detectable effects for ANOVA tables contains suspicious values in the region of large noncentrality parameter values compared to the values obtained by Patnaik's 2-moment central-*F* approximation. The reason is identified and the corrected form of the table for ANOVA purposes is given. The accuracy of the approximations to the noncentral-*F* distribution is also discussed.

Keywords: minimal detectable differences, ANOVA, noncentrality parameter, central-*F* approximations to noncentral *F*, recursive algorithms, symbolic computation

1. INTRODUCTION

Power, size of detectable effects and sample size calculations for ANOVA tables involve the computation of the noncentral *F*-distribution or its noncentrality parameter [see *e.g.* Johnson and Leone (1977)]. Tables, charts, and computer programs are available to perform these computations; furthermore, numerous approximations to the noncentral *F*-distribution exist. A thorough overview and several references can be found in chapter 30 of the book of Johnson, Kotz and Balakrishnan (1995), pp. 488-495. Subsequent computer programs were also published: Chattamvelli and Shanmugam (1997), Ding (1997), Tiwari and Yang (1997). Major modern statistical software packages offer this type of calculations.
The most useful and pioneering tables by Lorenzen and Anderson (1993, Appendix 12, pp. 374) give detectable differences for general ANOVA models belonging to a specified β probability of error of second kind, based on calculations with noncentral *F*-distribution. As some of the table entries seem to be suspicious, they have been compared to the values obtained by Patnaik's (1949) 2-moment central-*F* approximation and it is found that the table entries differ by orders of magnitude in the region of large noncentrality parameter values.
Many authors reported that one or more of the following problems can occur during the computations: underflow and overflow problems [Helstorm and Ritcey (1985), Ding (1997), Benton and Krishnamoorthy (2003)], round-off error that may introduce large relative error into the result [Frick (1990)], drastically increasing computation time and / or hanging up [Chattamvelli (1995), Benton and Krishnamoorthy (2003)], instability due to numerical cancellation [Knüsel and Bablok (1996)]. These facts motivated to scrutinize and find the critical points of the algorithms for computing the quantiles and the noncentrality parameter of the noncentral *F*-distribution as well as to study the accuracy of the approximations. Only sparse tables can be found for ANOVA purposes, often being obtainable with difficulty, so it is desirable to correct here Appendix 12 of Lorenzen and Anderson (1993), pp. 374. For simplicity, the latter table will be referred to as "the table".

The cdf $F(w,v_1,v_2,\lambda)$ of the noncentral *F*-distribution with v_1, v_2 degrees of freedom and noncentrality parameter λ and the cdf $I_x(a,b;\lambda)$ of the noncentral beta distribution with shape parameters a and b and noncetrality parameter λ are related by $F(w,v_1,v_2,\lambda) = I_x(a,b;\lambda)$, where $a = v_1/2$, $b = v_2/2$ and $x = v_1 w/(v_1 w + v_2)$ hold. Since easier to read formulae are yielded by the noncentral beta distribution, and the noncentral *F* distribution can be easily computed by using the above transformation of parameters, only the algorithm for the noncentral beta distribution is described. Furthermore, all three commonly used noncentral distributions (chi-square, *F* and *t*) can be computed in a manner analogous to that of the noncentral beta discussed below, similarly as proposed by *e.g.* Knüsel and Bablok (1996), Ding (1999), Benton and Krishnamoorthy (2003).

2. ALGORITHMS PROPOSED IN THE LITERATURE

2.1 Computing the cdf

The incomplete noncentral beta function ratio $I_x(a,b;\lambda)$ is defined as

$$I_x(a,b;\lambda) = \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} I_x(a+i,b), \tag{1}$$

where $I_x(a,b)$ is the usual incomplete beta function ratio,

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t,$$

$\Gamma(a)$ is the (complete) gamma function

$$\Gamma(a) = \int_0^{\infty} t^{a-1}e^{-t}\,dt;\ a > 0,$$

and $0 \le x \le 1$, $a > 0$, $b > 0$, $\lambda \ge 0$ [Johnson, Kotz and Balakrishnan (1995)].

The trivial cases $x = 0$, $x = 1$, $\lambda = 0$ are neglected. Let the Poisson multipliers in (1) be denoted by $p_i = e^{-\lambda/2}(\lambda/2)^i/i!$ (for i = 0, 1, …). Let ε denote a user defined small positive real number, *e.g.* $\varepsilon = 10^{-8}$ may be a good choice.

Let $k_1 = 0$ if $p_0 = e^{-\lambda/2} \ge \varepsilon$ else let k_1 be the greatest integer satisfying

$$\sum_{i=0}^{k_1-1} p_i \le \varepsilon, \qquad (k_1 \ge 1), \tag{2}$$

and let k_2 be the lowest integer satisfying

$$\sum_{i=k_2+1}^{\infty} p_i \le \varepsilon. \tag{3}$$

Performing the summation in (1) between indices k_1 and k_2 approximates $I_x(a,b;\lambda)$ with an absolute error less than 2ε, since

$$0 \le \sum_{i=0}^{\infty} p_i I_x(a+i,b) - \sum_{i=k_1}^{k_2} p_i I_x(a+i,b) = \tag{4}$$

$$= \sum_{i=0}^{k_1-1} p_i I_x(a+i,b) + \sum_{i=k_2+1}^{\infty} p_i I_x(a+i,b) \le \sum_{i=0}^{k_1-1} p_i + \sum_{i=k_2+1}^{\infty} p_i \le 2\varepsilon$$

2.1.1 Overflow and underflow problems

For large values of the noncentrality parameter, the first terms in (1) have to be omitted in order to avoid underflow (while computing $e^{-\lambda/2}$) or overflow (while computing $(\lambda/2)^i/i!$) problems, and to reduce the round-off error and computation time. For example, the algorithm of Norton (1983) or Lenth (1987) is exposed to over/underflow problems because they start the summation always at $i = 0$. It seems as if not omitting the first terms when necessary was the problem during the computation of Table 12 of Lorenzen and Anderson (1993) since the table entries corresponding to noncentrality parameters greater than 107.5 all correspond to exactly 107.5 whereas the other entries are correct up to at least 3 significant digits. (The authors regret to criticise an in all other aspects excellent book, from which they learned a lot.)

2.1.2. Cancellation problems in recursive algorithms

Lenth (1987) pointed out that significant reduction of the computation time is achievable if the following recurrence formulae are used to calculate the terms in (1) recursively

$$\frac{e^{-\lambda/2}(\lambda/2)^{i+1}}{(i+1)!} = \frac{(\lambda/2)}{i+1}\cdot\frac{e^{-\lambda/2}(\lambda/2)^{i}}{i!}, \qquad i = 0, 1, \ldots;$$

$$I_x(a+1,b) = I_x(a,b) - \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^b, \qquad \text{and} \qquad (5)$$

$$\Gamma(a+1) = a\Gamma(a).$$

Cancellation may occur if recurrence relationship (5) is used. According to Knüsel and Bablok (1996, pp. 1229), algorithms (namely the algorithm of Frick (1990) and Lenth (1987)) performing the summation in (1) forward ($i = k_1, k_1+1, \ldots, k_2$) are "instable due to numerical cancellation and the relative error cannot be kept under control". Wang and Kennedy (1995) also reported cancellation in the forward case (only partially in (5) and not severe). Avoiding cancellation problems can be guaranteed by using the following expression instead of equation (5)

$$I_x(a,b) = I_x(a+1,b) + \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^b \qquad (6)$$

and perform summation in (1) backward ($i = k_2, k_2-1, \ldots, k_1$), similarly to as proposed by Knüsel and Bablok (1996).

Numerical instability or severe cancellation has not been observed during the computation of the table using forward summation. Furthermore, the authors have not been able to create an example using practically reasonable parameters (roughly: cdf values = $10^{-3}\ldots1$, $\nu_1 = 1\ldots50$ $\nu_2 = 1\ldots1000$) where serious cancellation occurs. The incorrect values of the table of Lorenzen and Anderson seem to be caused by overflow error as already stated above.

2.1.3 Round-off error

In order to avoid the under/overflow problems caused by starting the summation always at $i = 0$ several authors suggest [Guirguis (1990), Posten (1993), Chattamvelli and Shanmugam (1997), Benton and Krishnamoorthy (2003)] starting from an index approximately equal to $\lambda/2$ and then work outward (increasing and decreasing *i*). Note that cancellation may occur whenever (5) is used in the forward iteration set. According to Henrici ((1982); pp. 8-10,

Demonstration 1.2-2), the outward summation is the worst way in a numerical sense because the terms to sum are (mainly) in decreasing order. The most accurate results could be achieved by the opposite inward technique at the expense of calculating two different starting terms (at k_1 and k_2), however the authors failed in finding such published algorithm.

Numerical tests using double-precision variables (64 bit) on a 32 bit machine have been carried out by the authors in order to study the difference between the outward and the inward technique. The relative error of the used auxiliary algorithms (not more than 8 correct significant digits) is several orders of magnitude greater than the round-off error introduced during the summation (less than 2000 iteration steps even for the largest noncentrality parameter values) and therefore totally covers up the difference mentioned. As a conclusion, the order of summation is a non-issue with the current desktop machines if double precision (64 bit) variables are used. However, it may be an issue with source codes using single-precision variables (32 bit or less), for example this may occur if a FORTRAN code with variables of type real is used (as downloaded from the Internet). *E.g.* the inaccurate values by Chattamvelli and Shanmugam (1997) in Table 2 of Section 4 seem to be caused by round-off error. Details of their computation are not known, however.

2.2 Computation of the noncentrality parameter

Given x, a, b and β, a numerical root-finding procedure is applied to solve

$$I_x(a,b,\lambda) = \beta \tag{7}$$

for λ.

2.2.1 Bisection and Newton iteration

The trivial cases $x = 0$ or $x = 1$ and / or $\lambda = 0$ are neglected. The following results can be derived from the book of Johnson, Kotz and Balakrishnan (1995) pp. 498 and / or Guirguis (1990). The derivative of the cdf with respect to λ is

$$\frac{d}{d\lambda} I_x(a,b;\lambda) = \frac{1}{2}\left(I_x(a+1,b;\lambda) - I_x(a,b;\lambda)\right). \tag{8}$$

The derivative in (8) is negative, so $I_x(a,b;\lambda)$ is a decreasing function of λ. There is no solution of (7) if $I_x(a,b) < \beta$ holds because $I_x(a,b;\lambda) \le I_x(a,b)$, and the trivial case $I_x(a,b) = \beta$ has the unique solution $\lambda = 0$. For a fixed x, a and b

$$\lim_{\lambda\to\infty} I_x(a,b;\lambda) = 0.$$

Thus, if $I_x(a,b) > \beta$ for given x, a, b, β then there is a unique solution of (7) for λ. If $I_x(a,b) > \beta$ holds and the root is bracketed [similarly to as proposed by Narula and Weistroffer (1986)] then the above-mentioned monotonicity property in λ guarantees the convergence of the bisection method.

The Newton iteration is usually more efficient than bisection. The 2-moment central-*F* approximation of Patnaik (1949) or the normal approximation of Severo and Zelen (1960) provides good starting values for the Newton iteration for the computation of the table, convergence is achieved in a few iteration steps for each entry.

2.2.2 Cancellation

Guirguis used directly the analogue of (8) in the noncentral *F* case to compute the derivative of the cdf with respect to λ, evaluating the noncentral *F* cdf twice in each Newton step. Cancellation can occur when applying (8) directly as done by Guirguis and as a consequence this may result in the unreliability of the method. As discussed in the next subsection, summation can be avoided by using recursive algorithms.

2.2.3 Reliability and efficiency issues with recursive algorithms

Ding (1997) pointed out that the computational efficiency of the Newton method can be greatly enhanced by eliminating the cdf evaluations in (8), and computing the derivative and the cdf simultaneously in the same iteration loop instead. This concurrent computation is realized by making use of the following rearrangement of (8) and (1)

$$\frac{d}{d\lambda} I_x(a,b;\lambda) = \frac{1}{2}\sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!}\left(I_x(a+1+i,b) - I_x(a+i,b)\right),$$

and the value of the factor $I_x(a+1+i,b) - I_x(a+i,b)$ has already been calculated for the recursive computation of incomplete beta function ratio dictated by (6) in the same iteration step as the Poisson weight (see also Step 5 in the proposed algorithm below). Note that this approach not only avoids the direct use of (8) and thus the cancellation as well, but it is also more efficient.

TABLE 1
Comparison of algorithms for the computation of noncentral *F* / beta distribution regarding reliability and efficiency issues

Algorithm	Exposed to over- or underflow	Exposed to cancellation[3]	Recursive	Iteration for lambda	Summation
Norton (1983)	yes	no	no	no	forward
Lenth (1987)	yes	yes	yes	no	forward
Frick (1990)[1] and Lam (1995)	no	yes	yes	no	forward
Guirguis (1990)	no	yes[4]	no	yes	outward[5]
Wang and Kennedy (1995)[2]	yes	yes	yes	no	forward
Chattamvelli and Shanmugam (1997)	no	yes	yes	no	outward[5]
Tiwari and Yang (1997)	yes	yes	yes	yes	forward
Ding (1997)	no	yes	yes	yes	forward
Proposed in Sec. 3	no	no	yes	yes	backward

1 Implemented in R (The R Project for Statistical Computing – free and multiplatform software environment)
2 Self-validating method
3 Backward summation and non-recursive algorithms for the cdf are not exposed to cancellation
4 Only for the derivative of the cdf with respect to lambda
5 The worst in numerical sense, can cause inaccuracy in single precision arithmetic

3. THE PROPOSED ALGORITHM

Given x, a, b and $prob$, the goal is to solve $I_x(a,b,\lambda) = prob$ for λ. The proposed algorithm is similar to the algorithm of Ding (1997) but considering that cancellation may occur using (5), the summation in (1) is carried out backward using (6). In order to avoid underflow and overflow errors, the first terms (i = 0, 1, …, $k_1 - 1$) are neglected if necessary. The trivial cases $x = 0$ or $x = 1$ and / or $\lambda = 0$ are not discussed, neither the case $I_x(a,b) < prob$ when there is no solution. Variables should be declared as double precision variables (at least 64 bit).

3.1. Auxiliary algorithms

The computation of the noncentral beta cdf according to (1) requires significant digit computation of the Poisson and the central beta distributions. The accuracy of the applied auxiliary algorithms obviously restricts the accuracy of the finally computed results. All auxiliary algorithms should use double-precision variables (at least 64 bit).

3.2. Input data

These data are x, a, b, λ_0, ε, $prob$, $N_{\max}$, δ. When solving $I_x(a,b;\lambda) = prob$ for λ, the values of x, a, b, $prob$ are known; an initial guess λ_0 must be obtained, *e.g.* the 2-moment central-*F* approximation of Patnaik (1949) or the normal approximation of Severo and Zelen (1960) is applicable. The user must specify the maximum tolerated absolute error of the cdf values during the inner computations, where ε is half of that error value. An absolute bound on the error of the derivative of the cdf with respect to λ is ε. The user specified δ is the maximum tolerated relative error of λ. The maximum number of the Newton iteration steps are $N_{\max}$, user specified.

3.3. Output datum

The output datum is the noncentrality parameter λ. When estimating the total error of the result both ε and δ must be considered, as well as the accuracy of the applied auxiliary algorithms.

3.4. Pseudo code

The assignment operator is denoted by "=" here (may be denoted by $\coloneqq$ or $\leftarrow$ in some other papers). $Beta(x,a,b)$ denotes the probability density function of the beta distribution with shape parameters a and b.

Step 1. $\lambda_{new} = \lambda_0$.

Step 2. Repeat steps 3 through 7 $N_{\max}$ times.

Step 3. Calculate the greatest k_1 and the lowest k_2 such that (2) and (3) hold.

Step 4. Initializations.

- Step 4.1. $k = k_2$
- Step 4.2. $\lambda = \lambda_{new}$
- Step 4.3. $\mu = \lambda / 2$
- Step 4.4. $c = I_x(a+k,b)$
- Step 4.5. $d = x(1-x)/(a+k-1)\cdot Beta(x,a+k-1,b)$
- Step 4.6. $p = e^{-\mu}\mu^k / k!$
- Step 4.7. $f = p\cdot c$
- Step 4.8. $p = k/\mu\cdot p$
- Step 4.9. $g = p\cdot d$

Step 5. For $k = k_2 - 1$, $k_2 - 2$, …, k_1, do steps 5.1 to 5.5.

- Step 5.1. $c = c + d$
- Step 5.2. $d = (a+k)/(x\cdot(a+k+b-1))\cdot d$
- Step 5.3. $f = f + p\cdot c$
- Step 5.4. $p = k/\mu\cdot p$
- Step 5.5. $g = g + p\cdot d$

Step 6. If $\lambda + 2(f - prob)/g \le 0$, then $\lambda_{new} = \lambda/2$; otherwise $\lambda_{new} = \lambda + 2(f - prob)/g$

Step 7. If $|\lambda_{new} - \lambda| < \delta \cdot \lambda$, then output λ_{new} and exit.

Step 8. Output the error message: 'Iteration did not converge in $N_{\max}$ steps'.

3.5. Notes

For computing the cdf only, skip steps 1, 2, 4.2, 4.9, 5.5, 6, 7, 8; the cdf value is the value of f obtained in the last evaluation of step 5.3.

The starting k_2 and stopping k_1 indices of the summation are computed in advance with an auxiliary algorithm, it is not the most effective way.

Note that the above algorithm can only guarantee an absolute bound ε on the error of the derivative (as $0 \le I_x(a+i,b) - I_x(a+i+1,b) \le 1$ holds, a similar estimation to (4) can be fulfilled). Fortunately, it did not cause instability or any other problem during the computation of the table.

4. FINITE EXPRESSION FOR COMPUTING THE CDF

For integer *b* values, Sibuya (1967) showed that

$$I_x(a,b;\lambda) = e^{-(\lambda/2)(1-x)} \sum_{i=0}^{b-1} \frac{\left((\lambda/2)(1-x)\right)^i}{i!} I_x(a+i,b-i), \tag{9}$$

and Singh and Relyea (1992) obtained

$$I_x(a,b) = x^a \left(1 + \sum_{n=1}^{b-1} \left(\prod_{m=1}^{n} \frac{a+m-1}{m} \right) (1-x)^n \right). \tag{10}$$

From (9) using (10) it follows that

$$I_x(a,b;\lambda) = e^{-(\lambda/2)(1-x)} \sum_{i=0}^{b-1} \frac{\left((\lambda/2)(1-x)\right)^i}{i!} \left(x^{a+i} \left(1 + \sum_{n=1}^{(b-i)-1} \left(\prod_{m=1}^{n} \frac{(a+i)+m-1}{m} \right) (1-x)^n \right) \right). \tag{11}$$

First, the finite and closed expression (11) is evaluated symbolically, and then arbitrary number of significant digits of the exact result can be computed (called "correct value" henceforth). To the authors' best knowledge this way has not been followed in the literature.

Numerical example

A table published by Chattamvelli and Shanmugam (1997) intended partly to prove that their algorithm gives more precise results than Frick's algorithm at large λ values. Their table is reproduced in Table 2 and supplemented by a column showing the correct results computed from equation (11). The proposed algorithm of Section 3 gives identical results to the correct values up to 7 significant digits.

The correct values contradict the conclusion of Chattamvelli and Shanmugam (1997) that their algorithm gives more precise results than Frick's algorithm at large λ values. If $\lambda \ge 54$, the algorithm of Chattamvelli and Shanmugam uses outward summation which is the worst in numerical sense, and may cause inaccuracy, especially when single-precision arithmetic is used (see also subsection 2.1.3). As Chattamvelli and Shanmugam give no sufficient details about their calculation, the reason of inaccuracy cannot be identified. On the other hand, there is no doubt about that their algorithm may be advantageous considering the terms of the number of iterations.

TABLE 2

***CDF of the non-central beta random variable*[1]**

a	*b*	λ	*x*	*Result for the following algorithms:*			
				Frick (1990)[2]	*Lenth (1987)*[2]	*Chatt. (1997)*[2,3]	*Correct*[4,5]
5	5	54	0.8640	0.4563026 (7)[1]	0.4563029 (6)	0.4563021 (5)	0.4563026
5	5	140	0.9000	0.1041342 (6)	0.1041331 (5)	0.1041337 (6)	0.1041335
5	5	170	0.9560	0.6022421 (6)	0.6022414 (5)	0.6022353 (5)	0.6022422
10	10	54	0.8686	0.9187790 (6)	0.9187794 (6)	0.9187770 (5)	0.9187791
10	10	140	0.9000	0.6008050 (5)	0.6008078 (5)	0.6008106 (5)	0.6008071
10	10	250	0.9000	0.0902795 (3)	0.0000000 (0)[6]	0.0902850 (4)	0.0902899
20	20	54	0.8787	0.9998677 (7)	0.9998677 (7)	0.9998655 (5)	0.9998677
20	20	140	0.9000	0.9925930 (4)	0.9925973 (5)	0.9925997 (5)	0.9925975
20	20	250	0.9220	0.9641169 (5)	0.0000000 (0)[6]	0.9641113 (4)	0.9641191

1 The number of correct significant digits is given in parentheses.
2 The values are from TABLE 1 of Chattamvelli and Shanmugam (1997).
3 Algorithm of Chattamvelli and Shanmugam (1997).
4 Result of the symbolic computations, given with 7 decimal places.
5 The proposed algorithm of Section 3 gives identical results.
6 Caused by underflow error, see subsection 2.1.1

The symbolic computations have been performed using Derive™ 5.04 (Texas Instruments). Calculation of the incomplete beta function ratio for integer *b* values according to (10) is far more effective than using the built-in INCOMPLETE_BETA function. Equation (11) can be evaluated not only symbolically but also using interval arithmetic [*e.g.* in C-XSC, see Klatte, Kulisch, Wiethoff, Lawo and Rauch (1993)], which may require less computation time and memory.

5. THE CORRECTED TABLE AND DISCUSSION OF APPROXIMATIONS

For computing the table, Newton method is applied to solve $I_x(a,b;\lambda)=\beta$ for λ, where $\beta=0.10$, x equals to the $\alpha=0.95$ quantile of the beta distribution with parameters $a=v_1/2$, $b=v_2/2$. The $\theta=\sqrt{\lambda/v_1}$ values are listed in the table as the function of v_1, v_2. Table 3 is the corrected form of Appendix 12 of Lorenzen and Anderson (1993), supplemented with an additional row, where $v_2=\infty$ [Johnson *et al.* (1995, pp. 487)]. This additional row is calculated with the Power Analysis module of STATISTICA 6.1 (StatSoft, Inc. 1984-2003). The rest of the table is calculated by the proposed algorithm coded in Statistica Visual Basic. Some cells of table have been checked with the above-mentioned finite and closed expression and also with the Power Analysis module of Statistica, there is good agreement (at least 6 significant digits).

TABLE 3
Table of minimal detectable differences[1]

Table of minimal detectable differences $\Delta = \sqrt{\Phi(Y)}/\sigma$ for a fixed factor or interaction *Y* in a general design; probability of the type I and type II errors are 0.05 and 0.10 respectively. Δ is obtained by dividing the table entry by $\sqrt{C}$, where *C* = coefficient in the expression of the expected mean square (EMS) preceding $\Phi(Y) = \sum_{i=1}^{r-1} \frac{\alpha_i^2}{r-1}$, with α_i the effect of factor *Y* at its *i*-th level, *r* the number of levels, σ^2 is the variance in the denominator of the *F* test statistic.[2]

	df numerator								
df denom.	1	2	3	4	5	6	10	20	50
1	20.96	23.25	24.16	24.65	24.95	25.15	25.57	25.89	26.08
2	6.796	6.710	6.682	6.668	6.659	6.653	6.642	6.633	6.628
3	5.014	4.630	4.475	4.390	4.336	4.299	4.221	4.159	4.121
4	4.396	3.900	3.692	3.576	3.502	3.450	3.339	3.250	3.193
5	4.092	3.538	3.301	3.166	3.079	3.018	2.886	2.777	2.707
6	3.913	3.324	3.068	2.921	2.825	2.757	2.609	2.486	2.405
7	3.795	3.183	2.914	2.759	2.656	2.583	2.423	2.287	2.197
8	3.712	3.084	2.805	2.643	2.535	2.458	2.288	2.142	2.044
10	3.604	2.953	2.661	2.489	2.375	2.292	2.107	1.944	1.832
12	3.536	2.871	2.570	2.392	2.272	2.186	1.989	1.814	1.690
14	3.489	2.815	2.508	2.325	2.202	2.112	1.907	1.721	1.588
16	3.455	2.774	2.463	2.276	2.150	2.058	1.846	1.652	1.510
18	3.429	2.743	2.428	2.239	2.111	2.017	1.800	1.598	1.449
20	3.409	2.718	2.401	2.210	2.080	1.984	1.762	1.554	1.399
22	3.393	2.698	2.379	2.186	2.054	1.957	1.732	1.519	1.357
24	3.380	2.682	2.361	2.166	2.033	1.935	1.707	1.489	1.322
26	3.368	2.669	2.346	2.150	2.016	1.917	1.686	1.464	1.292
28	3.359	2.657	2.333	2.136	2.001	1.901	1.667	1.442	1.265
30	3.351	2.647	2.322	2.124	1.988	1.888	1.652	1.423	1.242
40	3.322	2.613	2.283	2.082	1.944	1.841	1.597	1.355	1.159
60	3.295	2.580	2.246	2.042	1.900	1.794	1.542	1.287	1.070
80	3.281	2.563	2.227	2.022	1.878	1.772	1.515	1.252	1.022
100	3.273	2.554	2.216	2.010	1.866	1.758	1.498	1.231	0.9926
200	3.257	2.534	2.195	1.986	1.840	1.731	1.466	1.187	0.9298
500	3.248	2.523	2.182	1.972	1.825	1.715	1.446	1.161	0.8894
1000	3.245	2.519	2.178	1.967	1.820	1.709	1.439	1.152	0.8754
∞	3.242	2.515	2.173	1.962	1.815	1.704	1.433	1.143	0.8610

1 With reference to the incorrect values of the original table please see subsection 2.1.1
2 Please consult the book of Lorenzen and Anderson (1993) pp. 108 for illustrative examples.

Applicability of approximations

The bias caused by a particular approximation should be quantified by the realized type II error probabilities. Given v_1, v_2 and the $\alpha = 0.95$ quantile of the *F* distribution $F_{0.95}$, the equation $\tilde{F}(F_{0.95}, v_1, v_2, \tilde{\lambda}) = 0.10$ is solved for $\tilde{\lambda}$ with bisection, where $\tilde{F}$ denotes the approximating function to the cdf $F(w, v_1, v_2, \lambda)$ of noncentral *F* distribution (normal approximation of Severo and Zelen (1960), or 2-moment central-*F* approximation of Patnaik (1949), respectively; the corresponding formulae can be found in Johnson *et al.* (1995), pp. 491-495). Evidently, if $\tilde{F}$ was not an approximating function then $F(F_{0.95}, v_1, v_2, \tilde{\lambda})$ would yield the specified $\beta = 0.1000$ value. Table 4 shows the realized values of the type II errors if approximations are used. These results are in good agreement with Tiku's conclusion (1966). The accuracy of the studied approximations is satisfactory for ANOVA purposes.

TABLE 4
Realized type II error probabilities
The realized type II error probabilities if the noncentrality parameter is computed with approximation to the noncentral *F* distribution, as compared with the value 0.1 specified. Type I error probability is specified as 0.05.

	Approximation of Severo and Zelen (1960)				Approximation of Patnaik (1949)		
	df numerator				df numerator		
df denom.	1	5	50	df denom.	1	5	50
1	0.1042	0.1043	0.1043	1	0.1000	0.1000	0.1000
2	0.1024	0.1022	0.1023	2	0.1007	0.1000	0.1000
10	0.1125	0.1030	0.1005	10	0.1126	0.1028	0.1001
20	0.1158	0.1049	0.1004	20	0.1158	0.1048	0.1002
100	0.1187	0.1072	0.1007	100	0.1186	0.1070	0.1007
1000	0.1194	0.1079	0.1011	1000	0.1193	0.1077	0.1010

6. CONCLUSIONS

The studied recursive algorithms for computing the cdf and the noncentrality parameter of the noncentral *F* and noncentral beta distributions involve a certain summation. The major source of difficulties is that the first terms must be neglected if the noncentrality parameter is large otherwise underflow or overflow errors can occur. Backward recursion guarantees the avoidance of cancellation, however serious cancellation or instability has not been observed with forward recursion. The iterative computation of the noncentrality parameter using Newton method can be made more efficient, making use of the fact that the cdf and its first derivative can be computed simultaneously in the same iteration loop. Taking all these guidelines into consideration a new algorithm has been developed.
A finite and closed formula is derived for symbolic calculations. This formula enables studying the accuracy of different algorithms easily, an example is given in this paper.
The corrected form of the table for ANOVA purposes is given; it seems as if the problems with the original one were caused by overflow (or underflow) error.
Patnaik's 2-moment central-*F* approximation and the normal approximation by Severo and Zelen are compared: the accuracy of these approximations is satisfactory for ANOVA purposes. Both of these approximations provide good starting values for Newton's iterative computation of the noncentrality parameter.

References

Benton D. and Krishnamoorthy K. 2003. Computing discrete mixtures of continous distributions: noncentral chisquare, noncentral *t* and the distribution of the square of the sample multiple correlation coefficient, *Computational Statistics & Data Analysis*, **43**: 249-267.

Chattamvelli R. 1995. On the doubly noncentral *F* distribution, *Computational Statistics & Data Analysis*, **20**: 481-489.

Chattamvelli R. and Shanmugam R. 1997. Algorithm AS 310, Computing the Non-central Beta Distribution Function, *Applied Statistics*, **46**(1): 146-156.

Ding C. G. 1997. On using Newton's method for computing the noncentrality parameter of the noncentral *F* distribution, *Communications in Statistics – Simulation and Computation*, **26**(1): 259-268.

Ding C. G. 1999. An efficient algorithm for computing quantiles of the noncentral chi-squared distribution, *Computational Statistics & Data Analysis*, **29**: 253-259.

Frick H. 1990. AS R84. A remark on Algorithm AS 226, Computing noncentral beta probabilities, *Applied Statistics*, **39**: 311-312.

Guirguis G. H. 1990. A note on computing the noncentrality parameter of the noncentral *F* distribution, *Communications in Statistics – Simulation and Computation*, **19:** 1497-1511.

Helstorm C. W. and Ritcey J. A. 1985. Evaluation of the noncentral *F* distribution by numerical contour integration, *SIAM Journal on Scientific and Statistical Computing*, **6**(3): 505-514.

Henrici P. 1982. *Essentials of Numerical Analysis with Pocket Calculator Demonstrations*, New York [etc.], John Wiley & Sons.

Johnson N. L., Kotz S. and Balakrishnan, N. 1995. *Continuous Univariate Distributions* (Vol. 2., Second Edition), John Wiley & Sons, Inc., New York.

Johnson N. L. and Leone F. C. 1977. *Statistics and experimental design in Engineering and the Physical Sciences* (2nd ed.), J. Wiley and Sons, New York.

Klatte R., Kulisch U., Wiethoff A., Lawo C. and Rauch M. 1993. *C-XSC, A C++ Class Library for Extended Scientific Computing*, Springer-Verlag, Heidelberg, New York.

Knüsel L. and Bablok B. 1996. Computation of the noncentral gamma distribution. *SIAM Journal on Scientific Computing*, **17**(5): 1224-1231.

Lam, M. L. 1995. Remark AS R95: A remark on Algorithm AS 226: Computing Non-central Beta probabilities, *Applied Statistics*, **44**: 551-552.

Lenth R. V. 1987. Computing noncentral beta probabilities, *Applied Statistics*, **36**: 241-244.

Lorenzen T. J. and Anderson V. L. 1993. *Design of Experiments. A No-Name Approach.* Marcel Dekker, Inc.

Narula S. C. and Weistroffer H. R. 1986. Computation of probability and non-centrality parameter of noncentral *F* distribution, *Communications in Statistics – Simulation and Computation*, **15**: 871-878.

Norton V. 1983. A simple algorithm for computing the noncentral *F* distribution, *Applied Statistics*, **32**: 84-85.

Patnaik P. B. 1949. The non-central χ^2 and *F* distribution and their applications, *Biometrika*, **36**: 202-232.

Posten H. O. 1993. An effective algorithm for the noncentral beta distribution function, *The American Statistician*, **47**: 129-131.

Severo N. and Zelen M. 1960. Normal approximation to the chi-square and noncentral *F* probability functions, *Biometrika*, **47**: 411-416.

Sibuya M. 1967. On the noncentral beta distribution function, *unpublished manuscript*, the equation can be found in the book of Johnson, Kotz, and Balakrishnan (1995), pp. 485, Eq. (30.12).

Singh K. P. and Relyea G. E. 1992. Computation of noncentral *F* probabilities. A computer program, *Computational Statistics & Data Analysis*, **13**: 95-102. The misprint on pp. 97 was corrected by Chattamvelli (1995).

Tiku M. L. 1966. A note on approximating to the noncentral *F* distribution, *Biometrika*, **53**: 606-610.

Tiwari R. C. and Yang J. 1997. Algorithm AS 318: An efficient recursive algorithm for computing the distribution function and non-centrality parameter of the non-central *F*-distribution, *Applied Statistics*, **46**: 408-413.

Wang M. C. and Kennedy W. J. 1995. A self-validating numerical method for computation of central and non-central *F* probabilities and percentiles, *Statistics and Computing*, **5**: 155-163.