CS 170 Homework 9

Due **Monday 3/23/2026, at 10:00 pm (grace period until 11:59pm)**

Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, explicitly write "none".

1 Jeweler (Solo Question; 10 points)

You are a jeweler who sells necklaces and rings. Each necklace takes 4 ounces of gold and 2 diamonds to produce, each ring takes 1 ounce of gold and 3 diamonds to produce. You have 80 ounces of gold and 90 diamonds. You make a profit of 60 dollars per necklace you sell and 30 dollars per ring you sell, and want to figure out how many necklaces and rings to produce to maximize your profits.

(a) Formulate this problem as a linear programming problem. Draw the feasible region, and find the solution (state the cost function, linear constraints, and all vertices except for the origin).

(b) Suppose instead that the profit per necklace is C dollars and the profit per ring remains at 30 dollars. For each vertex you listed in the previous part, give the range of C values for which that vertex is the optimal solution.

Solution:

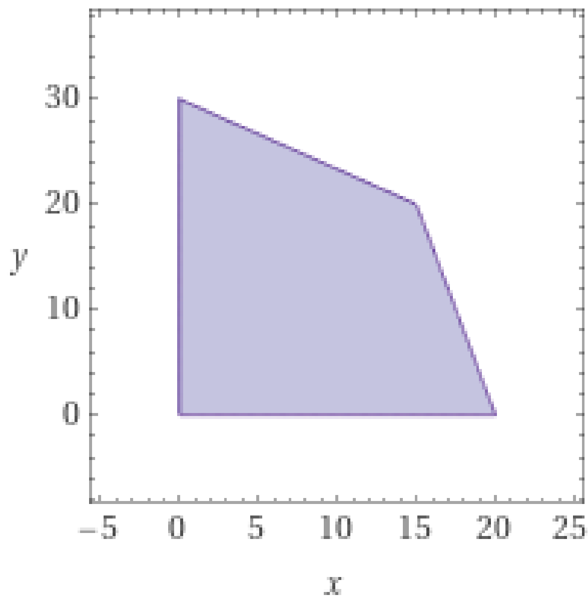
(a)
- Variables:

$$x = \text{number of necklaces}$$
$$y = \text{number of rings}$$

- Objective: Maximize $60x + 30y$.
- Linear constraints:

$$\begin{aligned} 4x + y &\leq 80 \\ 2x + 3y &\leq 90 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

- Feasible region:



- Vertices: $(x = 20, y = 0), (x = 15, y = 20), (x = 0, y = 30)$
- The objective is maximized at $(x = 15, y = 20)$, where $60x + 30y = 1500$

(b) Our objective is now $Cx + 30y$. The vertex $(0, 30)$ is optimal when C is small enough that the vertex $(15, 20)$ has a lower profit, that is when $C \cdot 15 + 30 \cdot 20 \geq 30 \cdot 30$, which simplifies to $C \geq 20$. The vertex $(15, 20)$ is optimal if $C \geq 20$ and if the vertex $(20, 0)$ has a lower profit, meaning that $C \cdot 15 + 30 \cdot 20 \geq C \cdot 20$, which simplifies to $C \leq 120$. Thus to summarize:

- $(0, 30)$: optimal for $C \leq 20$
- $(15, 20)$: optimal for $20 \leq C \leq 120$
- $(20, 0)$: optimal for $C \geq 120$.

2 Meal Replacement (10 points)

We are trying to eat cheaply but still meet our minimum dietary needs. We want to consume at least 500 calories of protein per day, 100 calories of carbs per day, and 400 calories of fat per day. We have three options for food we're considering buying: meat, bread, and protein shakes.

- We can consume meat, which costs 5 dollars per pound, and gives 500 calories of protein and 500 calories of fat per pound.
- We can consume bread, which costs 2 dollars per pound, and gives 50 calories of protein, 300 calories of carbs, and 25 calories of fat per pound.
- We can consume protein shakes, which cost 4 dollars per pound, and gives 300 calories of protein, 100 calories of carbs, and 200 calories of fat per pound.

Our goal is to find a combination of these options that meets our daily dietary needs while being as cheap as possible.

(a) Formulate this problem as a linear program.

(b) Take the dual of your LP from part (a).

(c) Suppose now there is a pharmacist trying to assign a price to three pills, with the hopes of getting us to buy these pills instead of food. Each pill provides exactly one of protein, carbs, and fiber.

Interpret the dual LP variables, objective, and constraints as an optimization problem from the pharmacist's perspective.

Solution:

(a) Let m be the pounds of meat we consume, b be the pounds of bread, and s be the pounds of shakes. The objective is to minimize cost, and we have a constraint for each of protein/carbs/fat:

$$\begin{aligned} \min\ 5m + 2b + 4s \\ 500m + 50b + 300s \geq 500 \\ 300b + 100s \geq 100 \\ 500m + 25b + 200s \geq 400 \\ m, b, s \geq 0 \end{aligned}$$

(b) Let p, c, f be variables corresponding to the protein, carb, and fat constraints. The dual is:

$$\begin{aligned} \max\ 500p + 100c + 400f \\ 500p + 500f \leq 5 \\ 50p + 300c + 25f \leq 2 \\ 300p + 100c + 200f \leq 4 \\ p, c, f \geq 0 \end{aligned}$$

(c) The variables can be interpreted as the price per calorie for the protein, carb, and fat pills.

The objective says that the pharmacist wants to maximize the total revenue they get from selling enough of these pills to us to meet our dietary needs.

The constraints say that no combination of pills should cost more than a pound of food providing the same dietary needs (otherwise, we would just buy that food instead of these pills).

3 Routing Data (10 points)

The internet is modelled as a directed network $G = (V, E)$, where the vertices are data centers, and edges represent connections between data centers. There are k types of data, and for the ith type of data, there is a source data center s_i, a destination data center t_i, and we want to transfer at least r_i units of this type of data through the network from s_i to t_i (we are allowed to transfer more). No other type $j \neq i$ of data can come from s_i or go to t_i. Using edge $e = (u, v)$, we can transfer at most c_e total units of data from u to v. For example, if $c_e = 3$, we could use e to transfer 3 units of type 1 data, or 1.5 units of each of type 1 and type 2 data (or any combination summing to ≤ 3). Our goal is to come up with a plan for routing each type of data, so that the total amount of data transferred is maximized.

Let $f_{i,e} \geq 0$ denote the number of units of the ith type of data we transfer using edge e. Write a linear program on these variables that captures the routing optimization problem described above. You should have constraints ensuring that each edge does not surpass its capacity, as well as that each vertex follows the requirements described above.

Solution:

- Objective: maximize the total amount of data coming from all the sources:

$$\max \sum_i \sum_{e=(s_i,v)} f_{i,e}.$$

- Constraint that each edge e has capacity c_e:

$$\sum_i f_{i,e} \leq c_e.$$

- Constraint that each source vertex s_i sends out $\geq r_i$ units of ith type data, and no other types of data:

$$\sum_{e=(s_i,v)} f_{i,e} \geq r_i$$
$$f_{j,e} = 0 \quad \text{for every } j \neq i \text{ and every } e = (s_i, v).$$

- Constraint that each target t_i receives only ith type data:

$$f_{j,e} = 0 \quad \text{for every } j \neq i \text{ and every } e = (v, s_i)$$

- Constraint that each vertex v that is not a source s_i or a target t_i has the same amount of incoming and outgoing data of each type i:

$$\sum_{e=(u,v)} f_{i,e} = \sum_{e=(v,w)} f_{i,e}.$$

4 Vertex Cover Dual (10 points)

In the vertex cover problem, we are given a graph $G = (V, E)$, and we want to find the smallest set of vertices S such that every edge has at least one vertex in S.

(a) Write an integer linear program (ILP) for this problem. That is, associate to each vertex v a variable $x_v \in \{0, 1\}$ indicating whether or not v lies in the set S. Then give the linear objective and constraints on these variables that capture the vertex cover problem.

(b) Take your ILP, and replace the constraints of the form $x_v \in \{0, 1\}$ with $x_v \geq 0$ to get a linear program (LP). Then find the dual LP of this LP. You can map this dual LP to an ILP by requiring every dual variable to lie in $\{0, 1\}$. What graph theory problem does this dual ILP represent? Give a natural graph-theoretic interpretation.

Solution:

(a) The primal problem on variables $x_v \in \{0, 1\}$ has:

- Objective: minimize $\sum_{v \in V} x_v$
- Constraints: $x_u + x_v \geq 1$ for every edge $(u, v) \in E$.

(b) If we relax the variable constraints to $x_v \geq 0$, then the dual problem has variables $y_e \geq 0$ for edges $e \in E$, with:

- Objective: maximize $\sum_{e \in E} y_e$
- Constraints: $\sum_{e \in E_v} y_e \leq 1$ for each vertex $v \in V$, where E_v denotes the set of edges incident to v.

If we restrict to integer variables $y_e \in \{0, 1\}$ in the dual, then the objective is the maximize the number of edges e with $y_e = 1$. The constraints say that no two such edges are allowed to share a vertex. This problem is exactly maximum matching.

5 Maximum Independent Set Dual (10 points)

Recall that an independent set of a graph is a set of vertices S such that no two vertices in S share an edge.

(a) Write an ILP (see the previous question) to find the maximum independent set in a graph.

(b) Replace every constraint of the form $x_v \in \{0, 1\}$ with $x_v \geq 0$ to get a LP, and find the dual LP. What problem does the dual represent? Again, create an ILP from the dual LP, and give a graph-theoretic interpretation.

Solution:

(a) The primal problem on variables $x_v \in \{0, 1\}$ has:

- Objective: maximize $\sum_{v \in V} x_v$
- Constraints: $x_u + x_v \leq 1$ for every edge $(u, v) \in E$

(b) If we relax the variable constraints to $x_v \geq 0$, then the dual problem has variables $y_e \geq 0$ for edges $e \in E$, with:

- Objective: minimize $\sum_{e \in E} y_e$
- Constraints: $\sum_{e \in E_v} y_e \geq 1$ for each vertex $v \in V$, where E_v denotes the set of edges incident to v.

If we restrict to integer variables $y_e \in \{0, 1\}$ in the dual, then the objective is to minimize the number of edges with $y_e = 1$. The constraints say that every vertex must be incident to such an edge. This problem is the minimum edge cover problem.

6 Integrality Gaps (10 points)

In the last two questions, we formulated natural graph theoretic problems as ILPs, and we found natural interpretations of their dual ILPs. However, the LP we obtain by "relaxing" the integer constraints $x \in \{0, 1\}$ can have non-integer solutions that beat all integer solutions. This phenomenon is captured by the *integrality gap*, defined as the ratio of the optimal solution of the ILP and the associated LP:

$$\text{Integrality Gap} = \frac{\text{OPT(ILP)}}{\text{OPT(LP)}}.$$

An integrality gap of > 1 (for minimization problems) or < 1 (for maximization problems) indicates that the optimal LP solution is non-integer.

(a) Give an example of a graph for which vertex cover has an integrality gap > 1. How large can you make the gap?

(b) Give an example of a graph for which maximum independent set has an integrality gap < 1. How small can you make it?

(c) Does the LP duality theorem still apply for ILPs? That is, given two ILPs whose LP relaxations are dual to each other, are the optimal ILP solutions equal?

If yes, explain why. If not, can you still see a use for LP duality in the context of these ILPs?

For concreteness, it may be helpful to think about vertex cover or maximum independent set.

Solution:

(a) If we take the complete graph on 3 vertices (i.e. the triangle graph), then the optimal ILP solution needs 2 of the 3 vertices to cover all edges. However, we have an LP solution with value $3/2$ by assigning value $x_v = 1/2$ to each vertex v. The integrality gap is $4/3 \approx 1.33$.

We can generalize this example to the complete graph on n vertices. The ILP solution needs $n - 1$ vertices to cover all edges, as if two vertices are not in our chosen set, the edge between them is not covered. However, we have an LP solution with value $n/2$ by again setting each $x_v = 1/2$. The integrality gap is $2(n - 1)/n$, which approaches 2 as $n \to \infty$.

(b) We again consider the complete graph on n vertices. The ILP solution has 1 vertex, as if we choose 2 vertices then the edge between them gives a violated constraint. But we have an LP solution with value $n/2$ by again setting each $x_v = 1/2$. The integrality gap is $2/n$, which approaches 0 as $n \to \infty$.

(c) The LP duality theorem no longer applies to ILPs, because it applies to LPs, and as we saw above, the ILP optimum can be worse than the LP optimum. Therefore there can be a gap between the optimal solutions of an ILP and the dual ILP.

However, the dual LP can still be used to certify bounds on the primal ILP (and vice versa). For instance, if we solve the dual LP to maximum independent set, we obtain an *upper bound* on the size of the maximum independent set. The integrality gap determines how tight this upper bound is.

