CS 170 Homework 9

Due **Monday 3/23/2026, at 10:00 pm (grace period until 11:59pm)**

Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, explicitly write "none".

1 Jeweler (Solo Question; 10 points)

You are a jeweler who sells necklaces and rings. Each necklace takes 4 ounces of gold and 2 diamonds to produce, each ring takes 1 ounce of gold and 3 diamonds to produce. You have 80 ounces of gold and 90 diamonds. You make a profit of 60 dollars per necklace you sell and 30 dollars per ring you sell, and want to figure out how many necklaces and rings to produce to maximize your profits.

(a) Formulate this problem as a linear programming problem. Draw the feasible region, and find the solution (state the cost function, linear constraints, and all vertices except for the origin).

(b) Suppose instead that the profit per necklace is C dollars and the profit per ring remains at 30 dollars. For each vertex you listed in the previous part, give the range of C values for which that vertex is the optimal solution.

2 Meal Replacement (10 points)

We are trying to eat cheaply but still meet our minimum dietary needs. We want to consume at least 500 calories of protein per day, 100 calories of carbs per day, and 400 calories of fat per day. We have three options for food we're considering buying: meat, bread, and protein shakes.

- We can consume meat, which costs 5 dollars per pound, and gives 500 calories of protein and 500 calories of fat per pound.
- We can consume bread, which costs 2 dollars per pound, and gives 50 calories of protein, 300 calories of carbs, and 25 calories of fat per pound.
- We can consume protein shakes, which cost 4 dollars per pound, and gives 300 calories of protein, 100 calories of carbs, and 200 calories of fat per pound.

Our goal is to find a combination of these options that meets our daily dietary needs while being as cheap as possible.

(a) Formulate this problem as a linear program.

(b) Take the dual of your LP from part (a).

(c) Suppose now there is a pharmacist trying to assign a price to three pills, with the hopes of getting us to buy these pills instead of food. Each pill provides exactly one of protein, carbs, and fat.

Interpret the dual LP variables, objective, and constraints as an optimization problem from the pharmacist's perspective.

3 Routing Data (10 points)

The internet is modelled as a directed network $G = (V, E)$, where the vertices are data centers, and edges represent connections between data centers. There are k types of data, and for the ith type of data, there is a source data center s_i, a destination data center t_i, and we want to transfer at least r_i units of this type of data through the network from s_i to t_i (we are allowed to transfer more). No other type $j \neq i$ of data can come from s_i or go to t_i. Using edge $e = (u, v)$, we can transfer at most c_e total units of data from u to v. For example, if $c_e = 3$, we could use e to transfer 3 units of type 1 data, or 1.5 units of each of type 1 and type 2 data (or any combination summing to ≤ 3). Our goal is to come up with a plan for routing each type of data, so that the total amount of data transferred is maximized.

Let $f_{i,e} \geq 0$ denote the number of units of the ith type of data we transfer using edge e. Write a linear program on these variables that captures the routing optimization problem described above. You should have constraints ensuring that each edge does not surpass its capacity, as well as that each vertex follows the requirements described above.

4 Vertex Cover Dual (10 points)

In the vertex cover problem, we are given a graph $G = (V, E)$, and we want to find the smallest set of vertices S such that every edge has at least one vertex in S.

(a) Write an integer linear program (ILP) for this problem. That is, associate to each vertex v a variable $x_v \in \{0, 1\}$ indicating whether or not v lies in the set S. Then give the linear objective and constraints on these variables that capture the vertex cover problem.

(b) Take your ILP, and replace the constraints of the form $x_v \in \{0, 1\}$ with $x_v \geq 0$ to get a linear program (LP). Then find the dual LP of this LP. You can map this dual LP to an ILP by requiring every dual variable to lie in $\{0, 1\}$. What graph theory problem does this dual ILP represent? Give a natural graph-theoretic interpretation.

5 Maximum Independent Set Dual (10 points)

Recall that an independent set of a graph is a set of vertices S such that no two vertices in S share an edge.

(a) Write an ILP (see the previous question) to find the maximum independent set in a graph.

(b) Replace every constraint of the form $x_v \in \{0, 1\}$ with $x_v \geq 0$ to get a LP, and find the dual LP. What problem does the dual represent? Again, create an ILP from the dual LP, and give a graph-theoretic interpretation.

6 Integrality Gaps (10 points)

In the last two questions, we formulated natural graph theoretic problems as ILPs, and we found natural interpretations of their dual ILPs. However, the LP we obtain by "relaxing" the integer constraints $x \in \{0, 1\}$ can have non-integer solutions that beat all integer solutions. This phenomenon is captured by the *integrality gap*, defined as the ratio of the optimal solution of the ILP and the associated LP:

$$\text{Integrality Gap} = \frac{\text{OPT(ILP)}}{\text{OPT(LP)}}.$$

An integrality gap of > 1 (for minimization problems) or < 1 (for maximization problems) indicates that the optimal LP solution is non-integer.

(a) Give an example of a graph for which vertex cover has an integrality gap > 1. How large can you make the gap?

(b) Give an example of a graph for which maximum independent set has an integrality gap < 1. How small can you make it?

(c) Does the LP duality theorem still apply for ILPs? That is, given two ILPs whose LP relaxations are dual to each other, are the optimal ILP solutions equal?

If yes, explain why. If not, can you still see a use for LP duality in the context of these ILPs?

For concreteness, it may be helpful to think about vertex cover or maximum independent set.