Aggregation of Heterogeneously Related Information with Extended Geometric Bonferroni Mean and Its Application in Group Decision Making

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Abstract

Capturing specific interrelationship among input arguments has great importance in the process of aggregation as they may change the aggregation result significantly, which can lead viable changes in the overall decision outcome. In this study, we attempt to aggregate a set of inputs with certain heterogeneous interrelationship pattern among them. To do this, we introduce a new aggregation operator which we call the extended geometric Bonferroni mean (EGBM). We investigate its properties and develop an algorithm to learn its associated parameters based on decision maker's perceived view towards the aggregation process. Moreover, to learn such heterogeneous relationship among the inputs from the data set, we provide a learning algorithm. Examples are given to illustrate the realization of algorithm and to show certain advantages over the existing aggregation operators.

Keywords: Aggregation function; Bonferroni mean; Extended Bonferroni Mean; Orness; Group decision making.

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1 Introduction

Aggregation of information coming from different sources plays a pivotal role in many engineering applications, such as pattern recognition, image analysis, rule-based systems, expert systems and decision making. Owing to its importance in practical applications, over the last decades aggregation functions have been studied extensively. In-depth accounts of aggregation functions and their properties can be found in the literature[1–3] with applications in science, engineering and economics.

Aggregation of information can be viewed as the process of combining several inputs into a single output[4] or a rule for producing a single piece of information from multiple pieces of information.[5] The output becomes a representative of all input information and provides an overall view about the system which essentially helps the decision maker to draw meaningful conclusions or take decisions.[6] For example, when we utilize the arithmetic mean to combine several experts' opinions (inputs) under the assumption that they are not influenced by each others, the output opinion represents the overall opinion of the group of experts an can be viewed as a collective opinion of the system. Here, the arithmetic mean is the rule for the aggregation system.

Modeling specific real-life scenarios and/or capturing various particular types of relations among input-output systems with the processing of different types of information, were the cornerstones behind the development of several classes of aggregation functions. For instance, ordered weighted average operator[7] allows us to give importance to the inputs according to their magnitudes and the weighted ordered weighted average operator[8] permits us to take account of the importance of information sources and magnitudes simultaneously. The prioritized aggregation operator[9] allows us to model the practical scenario of the multi-criteria decision making problem with prioritization among the criteria. The uninorm aggregation function[10] enables us to exhibit different behavior of input-output systems in different parts of the domain by choosing appropriate value of the neutral element $e \in [0, 1]$.

In this study, we concentrate on capturing a specific heterogeneous relationship structure among the aggregated inputs in the aggregation process. The Bonferroni mean is well known in the literature for capturing interrelationship among the aggregated arguments, after the works of Yager.[11] Its capability in modeling the mandatory requirements in the aggregation process and its generalization as an aggregation system were explored by Beliakov et al.[4] and others.[12–16] However, in most of the works related to BM, the basic interrelationship structure among the input set i.e., every input is related to the rest of the inputs, remains unaltered. In 2015, Dutta et al.[17] proposed an interesting extension of the Bonferroni mean, called the extended Bonferroni mean (EBM), for capturing heterogeneous interrelationships among the aggregated arguments, i.e., every input is related to a subset of the rest of the inputs.

In this study, we define another variant of the heterogeneous interrelationship capturing aggregation operator, which we refer to as the extended geometric Bonferroni mean operator (EGBM) to encompass in a single operator the advantages of the geometric mean and EBM. We explore its various properties and explain its working principle when specific heterogeneous interrelationships among inputs are given in advance. We illustrate via examples that semantically EBM and EGBM can model different types of mandatory requirements during the aggregation process. Another important aspect associated with this type of aggregation operator, from the application point of view, is how to select the associated parameters. We develop a learning algorithm to estimate the values of the parameters from sample observations based on a decision maker's perceived view towards aggregation. In addition, how to learn the said interrelationships among inputs from sample observations, when such heterogeneous relationships among the inputs is not given beforehand, is becoming an important aspect. We provide an algorithm based on similarity feature of inputs to learn such heterogeneous relationships from observations.

The rest of the paper is drafted as follows. In section 2, we define the extended geometric Bonferroni aggregation operator (EGBM) and consider several special cases based on the cardinality of the independent inputs. We also study the properties of EGBM and analyze the aggregated results by taking variation in the associated parameters. The advantage of EGBM over geometric mean and geometric Bonferroni mean is also explained via examples. In section 3, first we provide the necessary background of our learning principle and then realize the steps of the algorithm. A practical example is also provided to demonstrate the working of the algorithm. Section 4 describes the heterogeneous relationship learning algorithm. An example is also given to illustrate the realization of the algorithm. Section 5, presents our conclusions.

2 Extended Geometric Bonferroni Mean

Based on geometric mean and BM, Xia et al.[15] introduced the geometric Bonferroni mean (GBM) as follows:

Definition 1. [15] *Let* $\boldsymbol{a} = (a_1, a_2, ..., a_n)$*,* $n \geq 2$ *be a collection of non-negative real values. Assume* $p, q \geq 0$ *with* $p + q > 0$*. The GBM of the collection* $(a_1, a_2, ..., a_n)$ *is defined as:*

$$GBM^{p,q}(a_1, a_2, ..., a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{1}{n(n-1)}}. \tag{1}$$

Xia et al.[15] also showed that the GBM is an averaging aggregation function, i.e., it produces an aggregated value between the maximum and minimum values of the input set. Recently, Dutta et al.[17] proposed an interesting extension of the Bonferroni mean, called the extended Bonferroni mean (EBM), for capturing heterogeneous interrelations among the aggregated inputs. They classified the heterogeneously related inputs $A = (a_1, a_2, ..., a_n)$ into two disjoint subsets C and D, where each input a_i of C is related to a subset $B_i \subset C \setminus \{a_i\}$ and each of input of D is not related to any other inputs from $A \setminus \{a_i\}$. If I_i is the set of indices of the inputs of B_i and I' represents the indices of the input set D, then the aggregation rule for EBM is defined as follows:

Definition 2. [17] *For any* $p > 0$ *and* $q \geq 0$*, the EBM operator of dimension* n *is a mapping* $EBM : [0,1]^n \to [0,1]$ *such that*

$$EBM^{p,q}(a_1, a_2, ..., a_n) = \left(\frac{n - |I'|}{n} \left(\frac{1}{n - |I'|} \sum_{i \notin I'} a_i^p \left(\frac{1}{|I_i|} \sum_{j \in I_i} a_j^q \right) \right)^{\frac{p}{p+q}} + \frac{|I'|}{n} \left(\frac{1}{|I'|} \sum_{i \in I'} a_i^p \right) \right)^{\frac{1}{p}} \tag{2}$$

where $|I'|$ *denotes the cardinality of the set* I' *and empty sum is zero by convention with* $\frac{0}{0} = 0$*.*

EBM has a special capability of reflecting the expressed heterogeneous interrelationships among the aggregated arguments in the aggregation process, unlike the BM, which captures only a specific interrelationships among the aggregated arguments i.e., each input is related to the rest of the inputs. Apart from modeling the heterogeneous interrelationships among inputs, the EBM operator can model a particular requirement in the aggregation process which we will describe with help of an example. Consider one problem the following requirement: for producing a non-zero aggregated value, average of each pair of interrelated inputs must be above zero. For example, consider a trip location selection problem for a tour operator. There are two potential locations, denoted as $L1$ and $L2$, for the trip involving a group of six persons $\{P_1, P_2, P_3, P_4, P_5, P_6\}$. Among them (P_1, P_3), (P_2, P_4) and (P_5, P_6) are couples. That is they are interrelated in following fashion: P_1's choice depends on or related to P_3, P_2 's choice depends on or related to P_4 and P_5's choice depends on or related to P_6.

Owing to select the most suitable location, tour operator takes the satisfaction of each person against each of the locations and their satisfaction levels are summarized in Table I. Tour operator wants

Table I: Satisfaction level of the locations given the group

	P_1	P_2	P_3	P_4	P_5	P_6
L_1	0.8	0.6	0.5	0.8	0	0
L_2	0.4	0.3	0	0.9	0.5	0

to select a location by aggregating six individuals satisfaction levels, with additional requirement that average satisfaction level of each couple should be above zero level. This is the mandatory requirement seeking by tour operator in the aggregation process of finding locations' overall satisfaction levels. Now, we employ EBM operator to compute locations' overall satisfaction levels and, obtain $L1$'s score 0.29 and $L2$'s score 0.09. One may note that $L1$ is the best location according to overall satisfaction score even though the mandatory requirement for the couple (P_5, P_6) is not met. Even if the average satisfaction of each couple is above zero level for L_2, which ensures that the mandatory requirement is met, but L_2 is penalized by EBM. Clearly, EBM operator is unable to model the problem under the above mentioned mandatory requirement. In order to model such kind of decision scenarios, we look for a new aggregation operator which is to be illustrated in the following.

Based on the geometric mean and EBM, we propose a new aggregation operator for capturing heterogeneous relationships among the aggregated arguments, which we call the extended geometric Bonferroni mean (EGBM). Its main advantage over the use of EBM is to model mandatory requirement. It is defined as follows:

Definition 3. *For any $p, q \geq 0$ with $p + q > 0$, the extended geometric Bonferroni mean of dimension n*

is a mapping $EGBM : [0,1]^n \to [0,1]$ *such that*

$$EGBM_{p,q}(a_1, a_2, ..., a_n) = \frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+qa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i\in I'} a_i^{\frac{1}{|I'|}}\right) \tag{3}$$

where, we have made the convention empty product is 1. *The symbols and notations have the same interpretation as in Definition 2 with heterogeneous interrelationship pattern.*

Remark 1. *Depending on the context, one can interpret aggregated value by EGBM operator. Here, we interpret EGBM operator in the context of group opinion formation. Let* a_i *be the opinion of* i*-th individual against an alternative* x*. Then, we can interpret* $\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+qa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right)$ *as an opinion of subgroup of interrelated individuals and* $\prod_{i\in I'} a_i^{\frac{1}{|I'|}}$ *as an opinion of subgroup of independent individuals. Finally,* $EGBM_{p,q}(a_1, a_2, ..., a_n)$ *provides the opinion of the group by taking weighted average of the opinions of interrelated individuals and independent individuals.*

Remark 2. *It is to be noted that in Definition 3, we utilize weighted arithmetic mean to average aggregated value of interrelated inputs and independent inputs. However, one may use any averaging function* $f : [0,1]^2 \to [0,1]$ *instead of arithmetic mean.*

Based on the nature of the set I', the proposed EGBM operator is transformed into the three cases as follows:

(1) when $|I'| = n$, i.e., all the inputs are independent, we obtain geometric mean, independent of the parameters p and q as follows:

$$EGBM(a_1, a_2, ..., a_n) = \prod_{i=1}^{n} a_i^{\frac{1}{n}} \tag{4}$$

In this case, no interrelationship among the aggregated argument is captured.

(2) When $|I'| = 0$ and each input is related to the rest of the inputs, we recover the the geometric Bonferroni mean from Eq. (3) as follows:

$$EGBM_{p,q}(a_1, a_2, ..., a_n) = \frac{n-0}{n}\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+qa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-0}}$$

As each argument is related to rest of the arguments, $I_i = \{1, 2, ..., i-1, i+1, ..., n\}$ and $|I_i| = n-1$. It follows that

$$EGBM_{p,q}(a_1, a_2, ..., a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{1}{n(n-1)}};$$

(3) When $|I'| = 0$ and each input a_i is related to only a subset of the rest of inputs, we obtain

$$EGBM_{p,q}(a_1, a_2, ..., a_n) = \frac{1}{p+q} \prod_{i \notin I'} \left(\prod_{j \in I_i} (pa_i + qa_j)^{\frac{1}{|I_i|}} \right)^{\frac{1}{n-|I'|}}. \tag{5}$$

Now, we revisit the trip location selection problem and try to inspect the result produced by new aggregation method EGBM. From the description of interrelationship pattern, it is clear that $|I'| = 0$ as there is no independent persons in the group. Therefore, by employing Eq. (5), we compute the overall satisfaction score of the locations and obtain $L1$'s overall satisfaction score 0 and $L2$'s overall satisfaction score 0.13. Clearly, $L2$ is the best location for the trip to the group. Thus, for this selection the mandatory requirement is satisfied i.e., average satisfaction of each couple is above zero level. Thus, EGBM operator becomes an useful aggregation method and provides more intuitive and effective results when modeling mandatory requirement.

2.1 Properties of EGBM

We discuss the properties of EGBM below:

Theorem 1 (Idempotency)**.** *If all the inputs are equal, i.e., $a_i = a$ for all $i = 1, 2, ..., n$, then*

$$EGBM_{p,q}(a_1, a_2, ..., a_n) = a. \tag{6}$$

irrespective of parameters p, q and the interrelationship pattern among the inputs.

Proof: From Eq. (3), we have

$$
\begin{aligned}
EGBM_{p,q}(a,a,...,a) &= \frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa+qa)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right)+\frac{|I'|}{n}\left(\prod_{i\in I'}a^{\frac{1}{|I'|}}\right)\\
&= \frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left((pa+qa)^{\frac{1}{n-|I'|}}\right)\right)+\frac{|I'|}{n}a\\
&= \frac{n-|I'|}{n}\left(\frac{1}{p+q}(pa+qa)\right)+\frac{|I'|}{n}a\\
&= \frac{n-|I'|}{n}a+\frac{|I'|}{n}a\\
&= a.
\end{aligned}
$$

Theorem 2 (Commutative)**.** *Let $(b_1, b_2, ..., b_n)$ be the permutation of the input set $(a_1, a_2, ..., a_n)$. Then*

$$EGBM_{p,q}(a_1, a_2, ..., a_n) = EGBM_{p,q}(b_1, b_2, ..., b_n). \tag{7}$$

Proof: The values of the parameters p and q, and the interrelationship of each input with the other inputs remain intact in the permutation $(b_1, b_2, ..., b_n)$ as in the permutation $(a_1, a_2, ..., a_n)$, except the indices set I_i and I'. Therefore,

$$
\begin{aligned}
EGBM_{p,q}(a_1,a_2,...,a_n) &= \frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+qa_j)^{\frac{1}{|I'|}}\right)^{\frac{1}{n-|I'|}}\right)+\frac{|I_i|}{n}\left(\prod_{i\in I'}a_i^{\frac{1}{|I'|}}\right)\\
&= \frac{n-|\bar{I}'|}{n}\left(\frac{1}{p+q}\prod_{i\in\bar{I}'}\left(\prod_{j\in\bar{I}_i}(pb_i+qb_j)^{\frac{1}{|\bar{I}_i'|}}\right)^{\frac{1}{n-|\bar{I}'|}}\right)+\frac{|\bar{I}'|}{n}\left(\prod_{i\in\bar{I}'}b_i^{\frac{1}{|\bar{I}'|}}\right).
\end{aligned}
$$

where $\bar{I}'$ is the new indices set of independent arguments and $\bar{I}_i$ is the set of indices of the inputs which are related to b_i

Theorem 3 (Non-decreasing)**.** *Let $(a_1, a_2, ..., a_n)$ and $(c_1, c_2, ..., c_n)$ be the collection of two sets of inputs such that $a_i \leq c_i$ for all $i = 1, 2, ..., n$. If the heterogeneous relationship among the aggregated arguments is same for both input sets, then*

$$EGBM_{p,q}(a_1, a_2, ..., a_n) \leq EGBM_{p,q}(c_1, c_2, ..., c_n). \tag{8}$$

Proof: Since $a_i \leq c_i$ for all $i = 1, 2, ..., n$,

$$(pa_i + qa_j) \leq (pc_i + qc_j)$$

irrespective of the associated parameters p and q, and for all $i, j = 1, 2, ..., n$. This implies that,

$$\prod_{j \in I_i} (pa_i + qa_j)^{\frac{1}{|I_i|}} \leq \prod_{j \in I_i} (pc_i + qc_j)^{\frac{1}{|I_i|}} \quad \text{for each indices set } I_i$$

. It follows that

$$\prod_{i \notin I'} \left(\prod_{j \in I_i} (pa_i + qa_j)^{\frac{1}{|I_i|}} \right)^{\frac{1}{n-|I'|}} \leq \prod_{i \notin I'} \left(\prod_{j \in I_i} (pc_i + qc_j)^{\frac{1}{|I_i|}} \right)^{\frac{1}{n-|I'|}} \tag{9}$$

On the other hand, as $a_i \leq b_i$ for all $i = 1, 2, ..., n$

$$\frac{|I'|}{n} \left(\prod_{i \in I'} a_i^{1/|I'|} \right) \leq \frac{|I'|}{n} \left(\prod_{i \in I'} c_i^{1/|I'|} \right) \tag{10}$$

From Eqs. (9) and (10), we have

$$EGBM_{p,q}(a_1, a_2, ..., a_n) \leq EGBM_{p,q}(c_1, c_2, ..., c_n).$$

Theorem 4 (Boundedness)**.** *Let* $\max_i\{a_i\} = M$ *and* $\min_i\{a_i\} = m$*. Then,*

$$m \leq EGBM_{p,q}(a_1, a_2, ..., a_n) \leq M. \tag{11}$$

Proof: Boundedness directly follows from the non-decreasing (Theorem 3) and idempotency (Teorem 1) properties of $EGBM_{p,q}$.

Theorem 5 (Ratio scale invariant)**.** *The aggregation function* $EGBM : [0, 1]^n \to [0, 1]$ *is the ratio scale invariant, i.e., for any* $r > 0$*, we have*

$$EGBM_{p,q}(ra_1, ra_2, ..., ra_n) = rEGBM_{p,q}(a_1, a_2, ..., a_n) \tag{12}$$

Proof: From Eq. (3), we have

$$
\begin{aligned}
EGBM_{p,q}(ra_1, ra_2, ..., ra_n) &= \frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(p(ra_i)+q(ra_j))^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i\in I'}(ra_i)^{\frac{1}{|I'|}}\right) \\
&= r\frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+qa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + r\frac{|I'|}{n}\left(\prod_{i\in I'}(a_i)^{\frac{1}{|I'|}}\right) \\
&= rEGBM_{p,q}(a_1, a_2, ..., a_n).
\end{aligned}
$$

We adopt the following examples to explain the main characteristics of the EGBM operator.

Example 1. Let us consider the aggregation of the input set $\mathbf{a} = (a_1, a_2, a_3, a_4) = (0.5, 0.1, 0.4, 0.7)$ with the following interrelationships among inputs: a_1 is related to $\{a_2, a_3, a_4\}$, a_2 is related to $\{a_1, a_3\}$, a_3 is related to $\{a_1, a_2\}$ and a_4 is related to $\{a_1\}$. From the interrelationship pattern, it is clear that the indices sets are as follows: $I_1 = \{2, 3, 4\}$, $I_2 = \{1, 3\}$, $I_3 = \{1, 2\}$, $I_4 = \{1\}$ and $I' = \emptyset$. We aggregate the input set $\mathbf{a}$ with above heterogeneous interrelationship pattern and parameters $p = q = 1$. As $|I'| = 0$, with each input related to only to a subset of the rest of the inputs, we employ Eq. (5) to aggregate the inputs as follows:

$$
\begin{aligned}
EGBM_{1,1}(0.5, 0.1, 0.4, 0.7) = \frac{1}{2}\Bigg(&\left((0.5+0.1)^{1/3}\times(0.5+0.4)^{1/3}\times(0.5+0.7)^{1/3}\right)^{1/4} \\
&\times\left((0.1+0.5)^{1/2}\times(0.1+0.4)^{1/2}\right)^{1/4}\times\left((0.4+0.5)^{1/2}\times(0.4+0.1)^{1/2}\right)^{1/4} \\
&\times(0.7+0.5)^{1/4}\Bigg) \\
= 0.3504.
\end{aligned}
$$

It can be noted that the aggregated value lies between the minimum value of inputs 0.1 and maximum value of inputs 0.7. The aggregated value also differs from the values obtained by $GBM_{1,1}(0.5, 0.1, 0.4, 0.7) = 0.4056$ and geometric mean $GM(0.5, 0.1, 0.4, 0.7) = 0.3440$.

Example 2. We compare the result of Example 1 by considering different interrelationship patterns among the inputs as follows: a_1 is related to $\{a_3\}$, a_2 is related to $\{a_3, a_4\}$, a_3 is related to $\{a_1, a_2\}$ and a_4 is related to $\{a_2\}$. As earlier, using Eq. (5) with parameters $p = q = 1$, we obtain the aggregated value as follows:

$$EGBM_{1,1}(0.5, 0.1, 0.4, 0.7) = 0.3717.$$

It can be observed that the aggregated value is significantly different in comparison to Example 1, which indicates that the proposed aggregation operator reflects the interrelationship among the inputs in aggregated result. However, $GBM_{1,1}$ and GM provide the same aggregated value.

Now, we analyze the results of the Example 1 depending on the values of the parameters p and q.

Example 3. We compare the results of Example 1 by taking $(p, q) = (1, 0)$, $(p, q) = (2, 10)$, $(p, q) = (20, 1)$
$EGBM_{1,0}(0.5, 0.1, 0.4, 0.7) = 0.3440$, $EGBM_{2,10}(0.5, 0.1, 0.4, 0.7) = 0.3764$,
$EGBM_{20,1}(0.5, 0.1, 0.4, 0.7) = 0.3543$.

It can be seen that for the fixed interrelationship pattern among the inputs, the aggregation results depend on the choice of the parameters p and q. When $q = 0$, the aggregation result becomes equal to the result obtained by the GM. It can also be observed that for a fixed value of q (or p), the aggregated value by EGBM decreases with the increment of the parameter p (or q), and when p becomes large enough, the aggregation result by EGBM comes closer to the aggregated value produced by the geometric mean. The variation of the parameters p and q leads us to generate different aggregation results. However, $\max_i(a_i)$ and $\min_i(a_i)$ are never achievable via the choice of the parameters p and q from $[0, \infty)$ with $p + q > 0$. It signifies that a perfect conjunction or disjunction cannot be modeled via invoking the parameters p and q.

From Eq. (3) and the above examples, it is clear that EGBM generalizes GM and GBM, with the power of capturing user defined heterogeneous interrelationship pattern among the inputs. When the interrelationship pattern among the inputs is fixed, the combination of the parameters p and q allows us to model various levels of conjunction.

One may note that the averaging function EGBM is not monotone with respect to the parameters p and q. However, it is monotone in weaker sense.[18] That is illustrated in the following theorem:

Theorem 6. *For a fix $x \in [0, 1]^n$, the averaging function $EGBM_{p,q}$ is weakly monotone with respect to* *the parameters p and q, that is*

$$EGBM_{p+a,q+a}(x) \geq EGBM_{p,q}(x) \quad \text{for any } a \in \mathbb{R}. \tag{13}$$

Proof: To prove this, we utilize the fact that the weak monotonocity is equivalent to the non-negetivity of the directional derivative $D_1 EGBM_{p,q} \geq 0$. For that purpose, we first differentiate Eq. (3) with respect to p partially

$$\begin{aligned}\frac{\partial EGBM_{p,q}}{\partial p} &= \frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+qa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right)\left(\sum_{\substack{i\notin I'\\ j\in I_i\\ i\neq j}}\frac{q(a_i-a_j)}{(p+q)(pa_i+qa_j)|I_i|(n-|I'|)}\right)\\ &= L\sum_{\substack{i\notin I'\\ j\in I_i\\ j>i}}\frac{-(a_i-a_j)^2(p-q)q}{(p+q)(pa_i+qa_j)(pa_j+qa_i)|I_i|(n-|I'|)} \end{aligned} \tag{14}$$

where,

$$L = \frac{n-|I'|}{n}\left(\frac{1}{p+q}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+qa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right).$$

Similarly, differentiating both sides of Eq. (3) partially with respect to q, we obtain

$$\frac{\partial EGBM_{p,q}}{\partial q} = L\sum_{\substack{i\notin I'\\ j\in I_i\\ j>i}}\frac{(a_j-a_i)^2(p-q)p}{(p+q)(pa_i+qa_j)(pa_j+qa_i)|I_i|(n-|I'|)} \tag{15}$$

Now, we compute directional derivation of $EGBM_{p,q}$ in the direction $(1,1)$ by utilizing Eqs. (14) and (15) as follows:

$$\begin{aligned} D_1 EGBM_{p,q} &= \left(\frac{\partial EGBM_{p,q}}{\partial p}, \frac{\partial EGBM_{p,q}}{\partial q}\right).(1,1)\\ &= \frac{\partial EGBM_{p,q}}{\partial p} + \frac{\partial EGBM_{p,q}}{\partial q}\\ &= L\sum_{\substack{i\notin I'\\ j\in I_i\\ j>i}}\frac{(a_i-a_j)^2(p-q)^2}{(p+q)(pa_i+qa_j)(pa_j+qa_i)|I_i|(n-|I'|)}\\ &\geq 0 \end{aligned}$$

Hence the theorem.

Next, we analyze the proposed aggregation method by considering some special combinations of the parameters p and q.

2.2 Some special cases

Now, we consider some special cases of the parameters p and q:

- when $p = q$, the Eq. (3) becomes

$$EGBM_{p,p}(a_1, a_2, ..., a_n) = \frac{n-|I'|}{n}\left(\frac{1}{p+p}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i+pa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i\in I'} a_i^{\frac{1}{|I'|}}\right)$$
$$= \frac{n-|I'|}{n}\left(\prod_{i\notin I'}\left(\prod_{j\in I_i}(\frac{a_i+a_j}{2})^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I_i|}{n}\left(\prod_{i\in I'} a_i^{\frac{1}{|I'|}}\right)$$

  One can note that $EGBM_{p,p}$ becomes independent of parameter p. Therefore, in such case the choice of the parameters has no influence in the aggregation result.

- when $q = 0$, Eq. (3), becomes

$$EGBM_{p,0}(a_1, a_2, ..., a_n) = \frac{n-|I'|}{n}\left(\frac{1}{p+0}\prod_{i\notin I'}\left(\prod_{j\in I_i}(pa_i)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i\in I'} a_i^{\frac{1}{|I'|}}\right)$$
$$= \frac{n-|I'|}{n}\left(\frac{1}{p}\prod_{i\notin I'}(pa_i)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i\in I'} a_i^{\frac{1}{|I'|}}\right)$$
$$= \frac{n-|I'|}{n}\left(\prod_{i\notin I'} a_i^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i\in I'} a_i^{\frac{1}{|I'|}}\right)$$

  In this case no interrelationship among the dependent inputs is captured and parameter p has no effect in aggregation result and EGBM may be viewed as the weighted average of geometric mean of dependent inputs and geometric mean of independent inputs.

- When $p = 0$, the $EGBM_{p,q}$ operator is transformed into the following form

$$EGBM_{0,p}(a_1, a_2, ..., a_n) = \frac{n - |I'|}{n}\left(\frac{1}{0+q}\prod_{i \notin I'}\left(\prod_{j \in I_i}(qa_j)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i \in I'} a_i^{\frac{1}{|I'|}}\right)$$
$$= \frac{n - |I'|}{n}\left(\prod_{i \notin I'}\left(\prod_{j \in I_i} a_j^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i \in I'} a_i^{\frac{1}{|I'|}}\right)$$

- We can rewrite, Eq. (3) as

$$EGBM_{p,q}(a_1, a_2, ..., a_n) = \frac{n - |I'|}{n}\left(\prod_{i \notin I'}\left(\prod_{j \in I_i}\left(\frac{pa_i + qa_j}{p+q}\right)^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i \in I'} a_i^{\frac{1}{|I'|}}\right)$$

When p tends to ∞, the term $\frac{pa_i+qa_j}{p+q}$ tends to a_i and, thus,

$$\lim_{p \to \infty} EGBM_{p,q} = \frac{n - |I'|}{n}\left(\prod_{i \notin I'} a_i^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i \in I'} a_i^{\frac{1}{|I'|}}\right)$$

Clearly, it is equivalent to the aggregated value obtained by $EGBM_{p,q}$ when $q = 0$. Similarly, when q tends to ∞, we obtain

$$\lim_{q \to \infty} EGBM_{p,q} = \frac{n - |I'|}{n}\left(\prod_{i \notin I'}\left(\prod_{j \in I_i} a_j^{\frac{1}{|I_i|}}\right)^{\frac{1}{n-|I'|}}\right) + \frac{|I'|}{n}\left(\prod_{i \in I'} a_i^{\frac{1}{|I'|}}\right)$$

which is equivalent to the aggregated value obtained by $EGBM_{p,q}$ with $p = 0$. Hence, it is evident that for a large value of p (q), $EGBM_{p,q}$ behaves like the case when $q = 0$ ($p = 0$).

In the next section, we point out the advantages of employing EGBM in the aggregation process over popular aggregation operators: GM and the GBM.

2.3 Advantages

We have emphasized the fact of capturing the exact interrelationship among the aggregated inputs as they significantly affect the final choice. The main advantages of using EGBM to model mandatory criteria and semantically the difference of EGBM and EBM have been also highlighted. Now, we take a hypothetical example to show the advantage of using EGBM over GBM and GM in modeling heterogeneous relationship among the criteria. Consider the selection of the suitable alternative among options $\{X_1, X_2\}$ based on five factors $\{F_1, F_2, F_3, F_4, F_5\}$. The factors are heterogeneously interrelated and their interrelationships can be described as follows: F_1 is related to $\{F_2, F_3\}$, F_2 is related to $\{F_4, F_5\}$, F_3 is related to $\{F_1\}$, F_4 is related to $\{F_2\}$ and F_5 is related to $\{F_2\}$. Evaluation of the satisfaction of different options with respect to the factors are indicated in Table II. To select the best option, we need to compute overall satisfactions of the options by a suitable aggregation operator, and then compare them. If we employ the GM to compute the options' overall satisfaction, the problem of selecting the best option remains undecided as the geometric mean produces a zero overall satisfaction level for both the options. It mainly occurs as GM does not capture any kind of interrelationship among inputs. Next, we employ the GBM and the question of selecting the suitable alternative remains doubtful as GBM produces zero satisfaction level for both options irrespective of the fact that option X performs better under the factors F_1, F_2 and F_4. It mainly occurs due to the oversimplified assumption behind the working of GBM i.e., each input is related to the rest of the inputs, which enforces partial satisfaction of each pair of factors for non-zero scores. Now, we employ the EGBM for possessing the capability of modeling exact interrelationship among the aggregated inputs and obtain overall scores 0.3880 and 0.4819 for the options X_1 and X_2, respectively. Clearly, X_2 emerges as the best option. Therefore, EGBM shows certain advantage over the GM and GBM, when the inputs are heterogeneously related.

Table II: Satisfaction level of the alternatives under different factors

	F_1	F_2	F_3	F_4	F_5	GM	GBM	EGBM
X_1	0.8	0.6	0	0.2	0	0	0	0.3880
X_2	0.9	0.7	0	0.5	0	0	0	0.4819

When applying EGBM in practical application, one important issue is how to select the parameters p and q in particular decision scenarios. In the next section, we address this issue by developing a learning algorithm.

3 Learning the parameters p and q

We have seen that for a fixed interrelationship pattern among the aggregated arguments, the aggregation result by $EGBM_{p,q}$ depends on the parameters p and q. This variation in aggregation result may be linked with the local behavioral properties of the aggregation function, mainly *andness* and *orness* [1]. Fundamentally, *andness* measures the degree of similarity to which the aggregation operator is likely to be conjunctive (basically min operator) while *orness* measures the degree to which the aggregation operator is similar to disjunctive operator (basically, max operator). In practice, high *andness* makes satisfaction of all the input attributes more desirable and concedes a high penalty in the case of insufficient satisfaction of any of the input attributes. [19] High *orness* makes satisfaction for any of the input attributes more desirable, and the desired level of satisfaction of one input attribute is sufficient to satisfy a compound set of attributes. [19]

The measure of *orness* was first introduced as the disjunction degree for power means by Dujmovic [20]. Further, Yager [7] studied *orness* measure to investigate the properties of OWA operator. The extension of *orness* concept to the other aggregation operators has recently been studied by researchers [21–26]; and those studies are very important from both theoretical and application point of view. One of the most common application of *orness* measure is the weight vector determination of OWA operator under a given *ornes* level, which is usually formulated as a constrained optimization problem. [27–32] Inspired by this appealing point in the concept of *orness*, we exploit this local behavioral property of the aggregation function to learn the parameters p and q. One of the reasons for this is the fact that, in practice, the aggregation process is often subjective in nature and depends upon the context and decision maker's view on the relevant problem. A decision maker with a pessimistic view may prefer high *andness* while a decision maker with an optimistic perspective may prefer high *orness*. Before presenting the learning algorithm, we provide definitions of local *andness* and *orness* as follows:

Definition 4. [2] *The local andness of an aggregation function* $F : [0,1]^n \rightarrow [0,1]$ *is a function* laf_F *from* $[0,1]^n \setminus diag([0,1]^n) \rightarrow [0,1]$ *such that*

$$laf_F(\mathbf{x}) = \frac{\max(\mathbf{x}) - F(\mathbf{x})}{\max(\mathbf{x}) - \min(\mathbf{x})} \tag{16}$$

Definition 5. [2] *The local orness of an aggregation function* $F : [0,1]^n \rightarrow [0,1]$ *is a function* lof_F *from* $[0,1]^n \setminus diag([0,1]^n) \rightarrow [0,1]$ *such that*

$$lof_F(\mathbf{x}) = \frac{F(\mathbf{x}) - \min(\mathbf{x})}{\max(\mathbf{x}) - \min(\mathbf{x})} \tag{17}$$

As an immediate property, we find that $laf_F(\mathbf{x}) + lof_F(\mathbf{x}) = 1$.

The proposed algorithm makes an attempt to learn the parameters p and q associated with $EGBM_{p,q}$ on the basis of the decision maker's perspective towards aggregation, which can be expressed via *local orness*. In this sequel, we consider a real-life example in which we are given m sample observations, each of which is comprised of $n - tuple$ of non-negative real numbers $\mathbf{x_k} = (x_{k1}, x_{k2}, \cdots, x_{kn})(k = 1, 2, \cdots, m)$. As each of the sample is fused by $EGBM_{p,q}(\mathbf{x_k})$, each n-tuple is called the argument of the aggregation. We further assume that the arguments of each sample are heterogeneously related. In a typical MCDM, the arguments $(x_{k1}, x_{k2}, \cdots, x_{kn})$ interpreted as the rating of $\mathbf{x_k}$th alternative with respect to the n heterogeneously related attributes provided by the decision maker. The decision maker aggregates the arguments $(x_{k1}, x_{k2}, \cdots, x_{kn})$ to find the overall ratings of the alternative $\mathbf{x_k}$ by the aggregation operator $EGBM_{p,q}$ on the basis of which he/she will decide the best alternative.

Now, our intention is to aggregate each sample observation according to the decision maker's subjective view towards aggregation by learning the parameters p and q. The decision maker's view towards aggregation is expressed by the *local andness* measure α. We try to learn the parameters p and q under a given andness level α in such a way that

$$laf_{EGBM_{p,q}}(\mathbf{x_k}) = \frac{\max(\mathbf{x_k}) - EGBM_{p,q}(\mathbf{x_k})}{\max(\mathbf{x_k}) - \min(\mathbf{x_k})} = \alpha \tag{18}$$

satisfies at the best possible way for $k = 1, 2, ..., m$. For simplifying the notation, we assume that $laf_{EGBM_{p,q}}(\mathbf{x_k}) = \alpha_k$ for $k = 1, 2, ..., m$. In this sequel, the principle of our learning algorithm is to minimize the error between the decision maker's perceived view on aggregation i.e., α and the estimated view obtained from each sample i.e., α_k. Considering the mean square error, the parameters p and q can be learned by solving the following optimization problem

$$\begin{aligned} &\text{mainimize } \frac{1}{m}\sum_{k=1}^{m}(\alpha - \alpha_k)^2 \\ &\text{s.t. } \begin{cases} p \in [0, \infty) \\ q \in [0, \infty) \end{cases} \end{aligned} \tag{19}$$

Since, the objective function of the optimization problem Eq. (19) is a non-linear function of p and q, we require some effective algorithms to find the global optimal solution. Among different existing approaches for finding global solution of the non-linear optimization problem, evolutionary algorithmic (EA) approaches are widely used in practice. Therefore, we use the EA approach based solver to find the global optimizer. More precisely, we utilize the *ga* function implemented in the MATLAB optimization toolbox. This method finds the global optima of a non-linear function based on genetic algorithms.

Now, we formalize the steps of the learning parameters p and q as follows:

Algorithm-I: learning parameters p and q

Step 1 Input: set of observation, heterogeneous relationship among aggregated arguments.

Step 2 Set decision maker's view on aggregation by choosing appropriate value of $\alpha \in [0, 1]$

Step 3 Find the parameters p and q by solving the optimization model Eq. (19)

Now, we provide an example to illustrate the learning algorithm.

Example 4. Consider a random dataset which consists of 30 sample observations. Each observation is comprised of 5-tuple and signifies an alternative's degree of satisfaction under five performance determining factors $\{F_1, F_2, F_3, F_4, F_5\}$. The factors have a heterogeneous kind of interrelationship pattern and that has been depicted in Fig. 1. The satisfaction degree of each factor lies in the unit interval $[0, 1]$ and

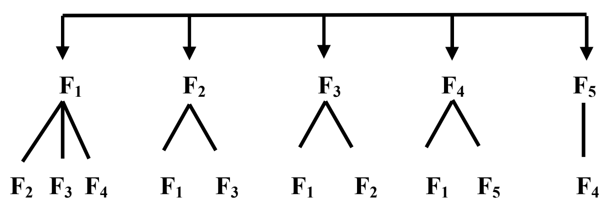

Figure 1: Heterogeneous interrelationship pattern among the factors

each factor is equally important. The dataset is presented in Table III.

In order to obtain each of the alternative's overall performance with respect to different factors under a decision maker's specified perceived view towards aggregation, we need to learn the parameters p and q of $EGBM$. The decision maker's perceived view towards aggregation is given via local *andness* measure degree 0.6. Now, we utilize the learning algorithm to estimate the values of p and q. The algorithm yields

Table III: Aggregated values of samples

Sample No.	F_1	F_2	F_3	F_4	F_5	aggregated value
1	0.8147	0.1622	0.6443	0.0596	0.4229	0.3153
2	0.9058	0.7943	0.3786	0.682	0.0942	0.4671
3	0.127	0.3112	0.8116	0.0424	0.5985	0.2547
4	0.9134	0.5285	0.5328	0.0714	0.4709	0.4077
5	0.6324	0.1656	0.3507	0.5216	0.6959	0.427
6	0.0975	0.602	0.939	0.0967	0.6999	0.3406
7	0.2785	0.263	0.8759	0.8181	0.6385	0.5134
8	0.5469	0.6541	0.5502	0.8175	0.0336	0.4
9	0.9575	0.6892	0.6225	0.7224	0.0688	0.4892
10	0.9649	0.7482	0.587	0.1499	0.3196	0.4678
11	0.1576	0.4505	0.2077	0.6596	0.5309	0.3525
12	0.9706	0.0838	0.3012	0.5186	0.6544	0.4025
13	0.9572	0.229	0.4709	0.973	0.4076	0.54
14	0.4854	0.9133	0.2305	0.649	0.82	0.5646
15	0.8003	0.1524	0.8443	0.8003	0.7184	0.5838
16	0.1419	0.8258	0.1948	0.4538	0.9686	0.4065
17	0.4218	0.5383	0.2259	0.4324	0.5313	0.4132
18	0.9157	0.9961	0.1707	0.8253	0.3251	0.5502
19	0.7922	0.0782	0.2277	0.0835	0.1056	0.177
20	0.9595	0.4427	0.4357	0.1332	0.611	0.4467
21	0.6557	0.1067	0.3111	0.1734	0.7788	0.3241
22	0.0357	0.9619	0.9234	0.3909	0.4235	0.3882
23	0.8491	0.0046	0.4302	0.8314	0.0908	0.2517
24	0.934	0.7749	0.1848	0.8034	0.2665	0.5089
25	0.6787	0.8173	0.9049	0.0605	0.1537	0.3532
26	0.7577	0.8687	0.9797	0.3993	0.281	0.5939
27	0.7431	0.0844	0.4389	0.5269	0.4401	0.3789
28	0.3922	0.3998	0.1111	0.4168	0.5271	0.3331
29	0.6555	0.2599	0.2581	0.6569	0.4574	0.4255
30	0.1712	0.8001	0.4087	0.628	0.8754	0.5045

the values of the parameters as follows: $p = 0.4813$ $q = 0.02$ with the mean square error 0.0094. The aggregated values of the samples with these learning parameters are presented in Table III. With the change of the decision maker's view towards aggregation via the *andness* measure, we obtain a different set of values for the parameters p and q with varying mean square error. Fig. 2 depicts the variation of mean square in the estimating the parameters p and q with decision maker's different attitudes.

We find that when decision maker's attitude is almost neutral or around neutral (i.e., andness around 0.5), the error in estimating the parameters becomes very low in comparison to the error when he/she takes an extremely optimistic or pessimistic attitude towards aggregation.

Remark 3. *It is to be noted that in learning the parameters p and q, we have utilized the local behavioral property (local* andness*) of EGBM. The measure of global* andness *involves multiple integral and the integral fold depends on the number of aggregated elements. When the number of aggregated arguments increases, the evaluation of multiple integral becomes complicated and so, exact analytical formula involv-*

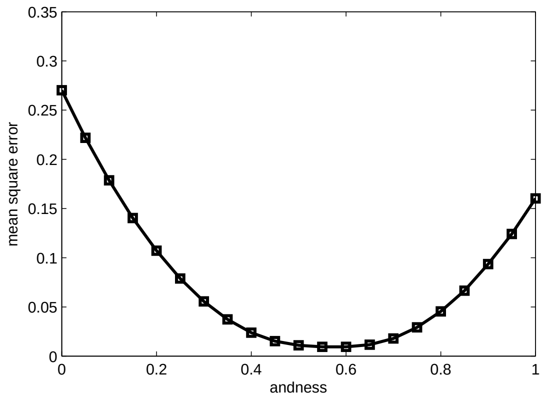

Figure 2: Variation in mean square error with andness measure

ing the parameters p and q for global andness *cannot be obtained. However, one can find the estimate of global* andness *by evaluating the associated multiple integral via* Monte Carlo *simulation for each pair of given values of the parameters p and q. In this way, we can find the distribution of global* andness *with respect to the parameters p and q. For doing this, the EGBM operator with heterogeneous interrelationship pattern as described in Fig. 1, can be put into following explicit mathematical form:*

$$EGBM_{p,q}(f_1, f_2, f_3, f_4, f_5) = \left(\frac{1}{p+q} \prod_{i \notin I'} \left(\prod_{j \in I_i} (pf_i + qf_j)^{\frac{1}{|I_i|}} \right)^{\frac{1}{5-|I'|}} \right) \tag{20}$$

where, $I' = \emptyset$, $I_1 = \{2, 3, 4\}$, $I_2 = \{1, 3\}$, $I_3 = \{1, 2\}$, $I_4 = \{1, 5\}$ *and* $I_5 = \{4\}$. *The distribution of global* andness *measure of the aggregation operator* $EGBM_{p,q}$ *(Eq. (20)) with respect to the parameters p and q has been depicted in Fig. 3.*

Another issue from application point of view is how to learn the interrelationships among the inputs from the dataset. In the next section, we develop an algorithm for the learning heterogeneous relationship among the arguments.

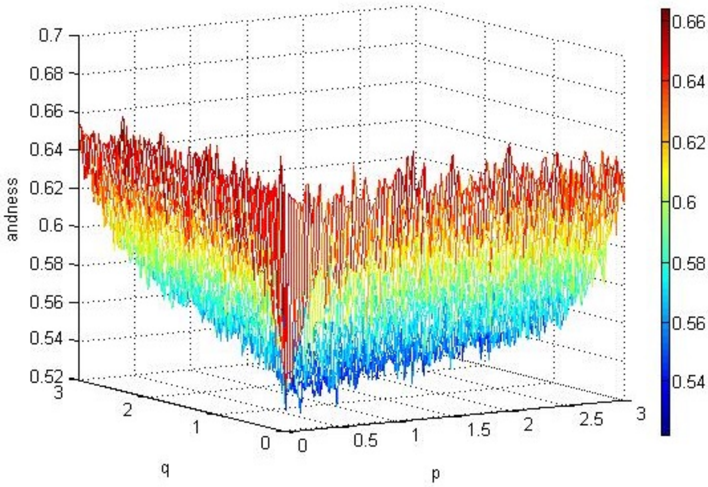

Figure 3: Distribution of global *andness*

4 Learning interrelationship pattern

An important issue in development of a new general decision theory is modeling the interaction among data. The main disadvantage of the existing theories is that different types of interrelationship pattern have not been fundamentally addressed. Therefore, the obtained results of decision analysis will naturally be partially reliable. Thus, in order to aggregate the result in the presence of interrelated data, modeling interrelationship is very important. So far, we have assumed that the heterogeneous relationships among the input arguments are in hand before the aggregation process starts. In this context, one important issue is that how to learn such kind of interrelationships among the input arguments from the sample observations. In this section, we propose a simple algorithm to learn the heterogeneous interrelationships from observation. Before presenting the algorithm, the semantics of heterogeneous interrelationship among the input arguments can be mathematized in the following way:

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be a set of input arguments. Then, the heterogeneous interrelationship among the input arguments can be expressed via matrix representation $HR = (\delta_{ij})_{n\times n}$, where the entry δ_{ij} is defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if } a_i \text{ is related to } a_j \\ 0 & \text{if } a_i \text{ is not related to } a_j \end{cases} \tag{21}$$

Example 5. Let us consider the input set $\mathbf{a} = (a_1, a_2, a_3, a_4)$ with the following interrelationship among inputs: a_1 is related to $\{a_2, a_3, a_4\}$, a_2 is related to $\{a_1, a_3\}$, a_3 is related to $\{a_1, a_2\}$ and a_4 is related to $\{a_1\}$. This heterogeneous interrelationship among the aggregated arguments can be expressed via matrix representation as follows:

$$HR = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Similarity is one of the important features that can be frequently used to learn interrelationships among data. We exploit this feature in our heterogeneous interrelationship learning algorithm. In this connection, we consider a practical example in which we are given m sample observations, each of which comprised of n-tuple non-negative real numbers $\mathbf{x_k} = (x_{k1}, x_{k2}, \cdots, x_{kn})(k = 1, 2, \cdots, m)$. As a typical example, let $\mathbf{x_k} = (x_{k1}, x_{k2}, \cdots, x_{kn})$ be the opinion of the n experts against the alternative $\mathbf{x_k}$. Now, we elaborate the steps of heterogeneous interrelationship learning algorithm as follows:

Step 1 Input m sample observations $\mathbf{x_k} = (x_{k1}, x_{k2}, \cdots, x_{kn})(k = 1, 2, \cdots, m)$.

Step 2 Compute the similarity among the input arguments of each sample. Let $\mathbf{x_k} = (x_{k1}, x_{k2}, \cdots, x_{kn})$ be any sample from m observations. The distance based similarity among any two input arguments x_{kj_1} and x_{kj_2} of $\mathbf{x_k}$ can be defined as

$$sm^k_{j_1 j_2} = 1 - |x_{kj_1} - x_{kj_2}| \tag{22}$$

Computing the similarity between each pair of input arguments of $\mathbf{x_k}$, we construct the similarity matrix $SM_k = (sm^k_{j_1 j_2})_{n\times n}$ associated with sample $\mathbf{x_k}$.

Step 3 Compute overall similarity among the input arguments of all samples. Let $SM = (sm_{j_1 j_2})_{n\times n}$ be the overall similarity matrix of the m samples in which each entry $sm_{j_1 j_2}$ is computed by using the arithmetic mean as follows:

$$sm_{j_1j_2} = \frac{1}{m}\sum_{k=1}^{m} sm^k_{j_1j_2} \tag{23}$$

Thus, $sm_{j_1j_2}$ will provide the overall similarity between the input argument x_{kj_1} and x_{kj_2} for all k

Step 4 From the overall similarity matrix, we construct the heterogeneous relationship matrix based on the threshold value of similarity $\eta \in [0, 1]$. If the similarity between two aggregated arguments is over the threshold η, we say the input arguments are interrelated, otherwise they have no interrelationship. In light of this fact, we construct the heterogeneous interrelationship matrix $HR = (\delta_{j_1j_2})_{n\times n}$ in which each entry δ_{ij} is defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if } sm_{j_1j_2} \geq \eta \text{ and } j_1 \neq j_2 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

The restriction $j_1 \neq j_2$ on indices j_1 and j_2 is imposed because of not considering the fact that input argument is related to itself. In other words, the reflexive relation among the input arguments is not considered here. From the Eq. (24), it is evident that the heterogeneous relationship pattern depends on the predefined threshold η. When $\eta = 0$, we obtain the heterogeneous relationship matrix $HR = (\delta_{j_1j_2})_{n\times n}$ with entries as 1, except the diagonal entries, which essentially implies that each input argument is related to the rest of the inputs. In case of distinct input arguments, the value of threshold $\eta = 1$ yields the heterogeneous interrelationship matrix HR as a zero matrix which signifies that there is no interrelationship among the input arguments.

Now, we provide an example to illustrate heterogeneous relationship learning algorithm.

Example 6. We consider a random dataset consists of ten sample observations. Each sample comprises of a 6-tuple which is basically opinions of 6-experts $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ against an alternative. To learn the interrelationship from the data, we implement the proposed algorithm. The dataset is presented in Table IV. The steps of the learning algorithm are realized as follows:

Step 1 Compute the similarity between decision makers' opinions for each sample and find the corresponding similarity matrix. For instance, the similarity matrix for sample $x_1 = (0.1239, 0.2085, 0.9479,$

Table IV: Sample data for learning relationship

Sample No.	E_1	E_2	E_3	E_4	E_5	E_6
1	0.1239	0.2085	0.9479	0.621	0.7378	0.8589
2	0.4904	0.565	0.0821	0.5737	0.0634	0.7856
3	0.853	0.6403	0.1057	0.0521	0.8604	0.5134
4	0.8739	0.417	0.142	0.9312	0.9344	0.1776
5	0.2703	0.206	0.1665	0.7287	0.9844	0.3986

$0.6210, 0.7378, 0.8589)$ can be computed by using Eq. (22) as follows:

$$SM_1 = \begin{pmatrix} 1 & 0.9154 & 0.176 & 0.5029 & 0.3861 & 0.265 \\ 0.9154 & 1 & 0.2606 & 0.5875 & 0.4707 & 0.3496 \\ 0.176 & 0.2606 & 1 & 0.6731 & 0.7899 & 0.911 \\ 0.5029 & 0.5875 & 0.6731 & 1 & 0.8832 & 0.7621 \\ 0.3861 & 0.4707 & 0.7899 & 0.8832 & 1 & 0.8789 \\ 0.265 & 0.3496 & 0.911 & 0.7621 & 0.8789 & 1 \end{pmatrix}$$

Similarly, we obtain similarity matrices for other samples $SM_k(k = 2, ..., 6)$.

Step 2 Utilizing Eq. (23) and similarity matrices $SM_k(k = 1, ..., 6)$, we compute the overall similarity among the opinions of the experts' overall alternatives as follows:

$$SM = \begin{pmatrix} 1 & 0.8214 & 0.4369 & 0.6206 & 0.6354 & 0.5611 \\ 0.8214 & 1 & 0.5857 & 0.5907 & 0.4906 & 0.714 \\ 0.4369 & 0.5857 & 1 & 0.5553 & 0.4812 & 0.7064 \\ 0.6206 & 0.5907 & 0.5553 & 1 & 0.6611 & 0.601 \\ 0.6354 & 0.4906 & 0.4812 & 0.6611 & 1 & 0.4934 \\ 0.5611 & 0.714 & 0.7064 & 0.601 & 0.4934 & 1 \end{pmatrix} \quad (25)$$

Step 3 We set the similarity threshold $\eta = 0.57$ and compute the heterogeneous interrelationship matrix with the help of Eq. (24) as follows:

$$HR = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Therefore, the heterogeneous interrelationship pattern among the experts becomes as follows: E_1 is related to $\{E_2, E_4, E_5\}$, E_2 is related to $\{E_1, E_3, E_4, E_6\}$, E_3 is related to $\{E_2, E_6\}$, E_4 is related to $\{E_1, E_2, E_5, E_6\}$, E_5 is related to $\{E_1, E_4\}$ and E_6 is related to $\{E_2, E_3, E_4\}$. Based on this derived interrelationship pattern, the $EGBM_{p,q}$ operator for aggregating this heterogeneously related information can take the following form:

$$EGBM_{p,q}(a_1, a_2, a_3, a_4, a_5, a_6) = \left(\frac{1}{p+q} \prod_{i \notin I'} \left(\prod_{j \in I_i} (pa_i + qa_j)^{\frac{1}{|I_i|}} \right)^{\frac{1}{6-|I'|}} \right) \tag{26}$$

where, $I' = \emptyset$, $I_1 = \{2, 4, 5\}$, $I_2 = \{1, 3, 4, 6\}$, $I_3 = \{2, 6\}$, $I_4 = \{1, 2, 5, 6\}$, $I_5 = \{1, 4\}$ and $I_6 = \{2, 3, 4\}$.

In this way, we can identify the heterogeneous interrelationship pattern from the sample dataset and the associated form of aggregation method, $EGBM_{p,q}$. By utilizing Eq. (26), the aggregated group opinion on each of the sample is obtained and the result is summarized in Table V. Here, the parameters p and q associated with $EGBM_{p,q}$ are taken as 1.

The similarity threshold η plays a crucial role in determining the heterogeneous interrelationship pattern. In fact a small change of the threshold value can significantly alter the heterogeneous interrelationship pattern. For instance, if we change the similarity threshold η from 0.57 to 0.6, the new

Table V: Group opinion on each of the sample

Sample No.	Aggregated Results
1	0.5056
2	0.4384
3	0.4603
4	0.4979
5	0.4017

heterogeneous interrelationship matrix becomes

$$HR = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

It is quite visible from the new heterogeneous interrelationship matrix that the interrelationship pattern for experts E_2, E_3 and E_4 have changed. They are now interrelated to a fewer numbers of expert opinions in comparison to the earlier case. On the basis of this new interrelation pattern, the proposed aggregation method, $EGBM_{p,q}$ can be represented as follows:

$$EGBM_{p,q}(a_1, a_2, a_3, a_4, a_5, a_6) = \left(\frac{1}{p+q} \prod_{i \notin I'} \left(\prod_{j \in I_i} (pa_i + qa_j)^{\frac{1}{|I_i|}} \right)^{\frac{1}{5-|I'|}} \right) \tag{27}$$

where, $I' = \{1, 2, 3, 4, 5\}$, $I_1 = \{2, 4, 5\}$, $\mathbf{I_2 = \{1, 6\}}$, $\mathbf{I_3 = \{6\}}$, $\mathbf{I_4 = \{1, 5, 6\}}$, $I_5 = \{1, 4\}$ and $I_6 = \{2, 3, 4\}$. The bold face letters indicate the changes from Eq. (26). With this new form of $EGBM_{p,q}$, we compute the group opinion against each of the sample and the results are summarized in Table VI.

From Table VI, it is observed that overall group opinion against each sample has changed significantly from earlier group opinion (Table V). Therefore, the threshold parameter η might be highly sensitive and

Table VI: Group opinion on each of the sample

Sample No.	Aggregated Results
1	0.5102
2	0.4588
3	0.4788
4	0.4784
5	0.4175

context dependent. Moreover, the high value of η can help us to interrelate the experts with a high degree of similarity in their opinions, and η can be viewed as the intensity of the heterogeneous interrelationship pattern.

5 Application

In this section we provide an application of the proposed aggregation technique in group decision making.

An university administration has decided to give promotion to the faculty members of a department. The administration agrees upon the fact that only performances of candidates will be considered as a sole basis for promotion. Three attributes: teaching, research and service are set as the determinant to measure the performances. After initial screening, eight faculty members $\{Y_1, Y_2, ..., Y_8\}$ are considered for promotion. For final evaluation, administration has set up a committee which consists of ten experts $\{E_1, E_2, ..., E_{10}\}$. On the basis of the performance measuring attributes, experts provide their ratings for each candidate on scale $[0, 1]$. The ratings provided by the experts are summarized in Table VII. Our aim is to form the group overall ratings against each of the candidate and rank them.

Table VII: Ratings of the experts

Faculty.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
1	0.80	0.90	0.85	0.80	0.55	0.55	0.50	0.55	0.70	0.85
2	0.90	1.00	0.95	0.85	0.60	0.60	0.55	0.65	0.75	0.95
3	0.70	0.90	0.80	0.75	0.55	0.45	0.55	0.50	0.90	0.90
4	0.85	0.95	0.90	0.90	0.70	0.60	0.60	0.60	0.95	1.00
5	0.80	0.85	0.85	0.85	0.75	0.80	0.75	0.70	0.95	0.90
6	1.00	0.95	0.95	0.95	0.85	0.90	0.85	0.85	1.00	1.00
7	0.60	0.90	0.90	0.85	0.85	0.85	0.80	0.70	0.60	0.55
8	0.80	0.95	1.00	0.90	0.90	0.90	0.85	0.80	0.70	0.60

The experts might have interrelationship in the sense that one can influence others' decisions and that can affect the ratings of a candidate. As the capturing interrelationship among the arguments in the aggregation process affects the final decisions significantly, we need to identify such kind of interrelationship pattern before the formation of group opinion against each of the candidate. In order to learn the heterogeneous interrelationship among the experts from the dataset given in Table VII, we employ the heterogeneous relationship learning algorithm described in Section 4. Taking the similarity threshold value $\eta = 0.85$, we obtain the heterogeneous relationship matrix as follow:

$$HR = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \\ E_9 \\ E_{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Another issue in the formation of group opinions is capturing group attitude towards aggregation of individual opinions. The attitude of group can be incorporated in the aggregation process via *local andness* measure as discussed earlier.[31,32] Based on the heterogeneous interrelationship pattern and group attitude (given by local andness $\alpha = 0.4$), we learn the parameters p and q associated with $EGBM_{p,q}$ operator by utilizing the learning algorithm, described in Section 3, and obtain the values of the parameters as follows: $p = 0.4055$ and $q = 2.4190$. Now, based on the heterogeneous interrelationship pattern (HR) with the parameters $p = 0.4055$ and $q = 2.4190$, the aggregation operator $EGBM_{p,q}$ for finding the overall group ratings against each of the candidates takes the following form:

$$EGBM_{p,q}(e_1, e_2, ..., e_{10}) = \left(\frac{1}{0.4055 + 2.4190} \prod_{i \notin I'} \left(\prod_{j \in I_i} (0.4055 e_i + 2.4190 e_j)^{\frac{1}{|I_i|}} \right)^{\frac{1}{5-|I'|}} \right) \quad (28)$$

where, $I' = \{1, 2, ..., 10\}$, $I_1 = \{2, 3, 4, 9, 10\}$, $I_2 = \{1, 3, 4, 9, 10\}$, $I_3 = \{1, 214, 10\}$, $I_4 = \{1, , 2, 3, 5, 6, 9, 10\}$, $I_5 = \{4, 6, 7, 8\}$, $I_6 = \{4, 5, 7, 8\}$, $I_7 = \{5, 6, 9\}$, $I_8 = \{5, 6, 7\}$, $I_9 = \{1, 2, 4, 10\}$ and $I_{10} = \{4, 5, 7, 8\}$.

By utilizing Eq. (28), we obtain overall ratings of the each candidate. The aggregation results has been summarized in Table VIII with the rank of the alternatives.

Table VIII: Ratings of the experts

Faculty.	Overall group ratings	Rank
1	0.6975	7
2	0.7700	5
3	0.6816	8
4	0.7961	4
5	0.8182	3
6	0.9296	1
7	0.7570	6
8	0.8372	2

6 Conclusions

We have presented a composed aggregation operator by utilizing the core of the extended Bonferroni mean and geometric mean, and called the extended geometric Bonferroni mean (EGBM). This operator captures the heterogeneous relationship among the input arguments with some additional requirements in the process of aggregation. This new aggregation operator can take advantages of the geometric Bonferroni mean and geometric mean in modelling heterogeneous interrelationship structures. We have studied the property of the EGBM operator in detail. From the application perspective, two important issues associated with this proposed aggregation operator are how to learn the parameters associated with it and how to learn such specific heterogeneous interrelationships from the data-set. We have addressed those issues. We have developed an algorithm to learn the associated parameters by fixing the decision maker's view towards the whole aggregation process when heterogeneous interrelationships among the inputs are fixed. To learn the interrelationship from data, we have also proposed a learning algorithm based on similarity features of data. The proposed aggregation approach has specific advantages which are listed below:

- the proposed aggregation technique can fuse the information according to user defined heterogeneously related information and can reflect the effect of the interrelationship pattern in the aggregated value in a more viable way than other aggregation operators, such as the geometric mean, geometric Bonferroni mean.
- by adjusting the parameters p and q, a decision maker's perceived view towards aggregation can be incorporated in the process of information fusion
- when interrelationship pattern among the inputs is not predefined, however, a decision maker anticipates that there exists some sort of interrelationship among the inputs, in such cases the heterogeneous relationship learning algorithm works as a guiding tool in identifying the relationship among the inputs on the basis of decision information before the aggregation process takes place.

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